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Mathematical Reviews

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
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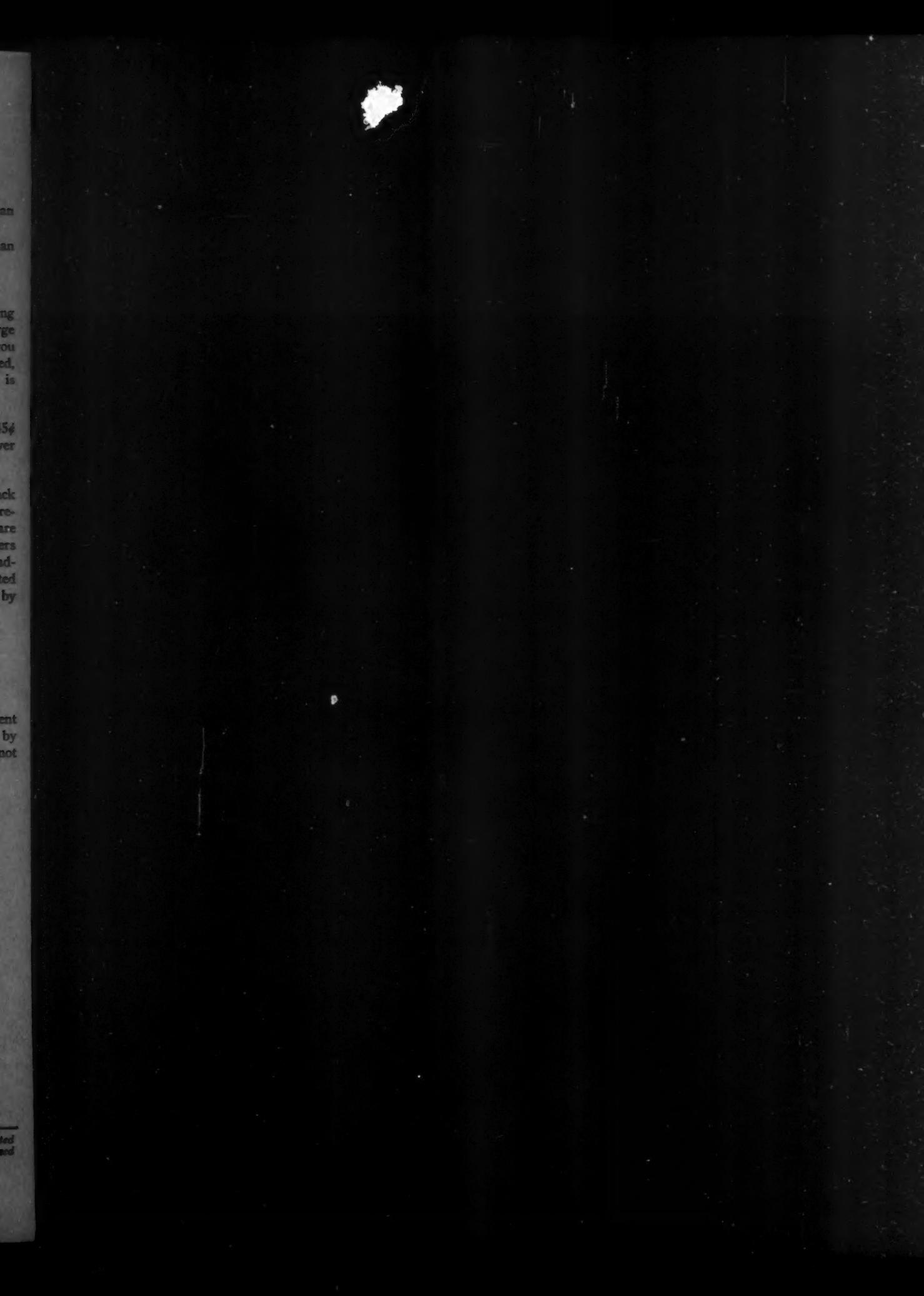
Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.



Mathematical Reviews

Vol. 4, No. 4 APRIL, 1943 Pages 97-124

ANALYSIS

Fourier Series and Generalizations, Integral Transforms

Pólya, George and Wiener, Norbert. On the oscillation of the derivatives of a periodic function. Trans. Amer. Math. Soc. **52**, 249–256 (1942). [MF 7119]

Let $f(x)$ be a real valued function of period 2π, having derivatives of all orders for all real x. Let N_k denote the number of changes of sign of $f^{(k)}(x)$ in a period. If, as $k\to\infty$, $N_k=O(1)$, $f(x)$ is a trigonometric polynomial; if $N_k=o(k^k)$, $f(x)$ is an entire function; if $N_k=O(k^\delta)$, $0<\delta<\frac{1}{2}$, $f(x)$ is an entire function of order not exceeding $(1-\delta)/(1-2\delta)$. The proof depends on the fact that $\sum c_n e^{inx}$ has at least as many changes of sign in a period as $\sum c_n(a^2+n^2)^{-1}e^{inx}$ (a a real constant). [By quite different methods, A. C. Schaeffer has recently shown [Proc. Nat. Acad. Sci. U.S.A. **28**, 62–64 (1942); these Rev. **3**, 144] that a real valued function, not necessarily periodic, is analytic if $N_k=O(1)$.] *E. S. Pondiczery* (Princeton, N. J.).

Szegö, G. On the oscillation of differential transforms. I. Trans. Amer. Math. Soc. **52**, 450–462 (1942). [MF 7456]

Let $f(x)$ be a function of period 2π, differentiable infinitely many times, and let $2N_k$ denote the (necessarily even) number of the sign variations of $f^{(k)}(x)$ in the interval $0\leqq x<2\pi$. Pólya and Wiener showed [cf. the paper reviewed above] that, if $N_k\leqq N$ for infinitely many values of k, then $f(x)$ is a trigonometric polynomial of order $\leqq N$. The present paper gives a new proof, based on Parseval's formula, of this theorem. The new argument permits of an improvement of other Pólya-Wiener results. In particular: (1) If $N_k\leqq k/\log k$ for all k sufficiently large, then $f(x)$ is an entire function; (2) if $N_k\leqq\frac{1}{2}(k/\rho)^{1/\rho}$ for some $\rho>1$ and all k sufficiently large, $f(x)$ is an entire function of order $\leqq\rho/(\rho-1)$. The paper concludes with generalizations when $f^{(2)}(x)$ is replaced by the differential operator $\vartheta^1(x)$, where

$$\vartheta=(1-x^2)D^2-2xD, \qquad D=d/dx.$$

A. Zygmund (South Hadley, Mass.).

Hille, Einar. On the oscillation of differential transforms. II. Characteristic series of boundary value problems. Trans. Amer. Math. Soc. **52**, 463–497 (1942). [MF 7457]

[For the first part cf. the preceding review.] The paper gives extensions of the important results of Pólya and Wiener [cf. the review preceding the one above] to the case when the operator $D^2(=d^2/dx^2)$ is replaced by a rather general differential operator L of the second order. Introducing certain boundary value conditions at the endpoints of the basic interval (a, b), we may consider the "characteristic series" (*) $\sum f_n u_n(x)$ of L, where the coefficients f_n are constants and the u_n are "characteristic functions" of L, that is, satisfy the boundary value conditions and the relations $Lu_n(x)=\lambda_n u_n(x)$, the λ's being the "characteristic constants." The main result of the paper is that, if the operator L and the boundary value conditions have certain properties, if a function f is represented by a characteristic series (*) which converges so rapidly that the operation L may termwise be applied to it as many times as we wish, then the condition that for infinitely many values of k the number of sign-changes of $L^k f(x)$ does not exceed N implies that $f(x)$ is a linear combination of a finite number of the u_n, none of which may have more than N (exceptionally, $N+1$) changes of sign in the interval (a, b). For the precise statement of the assumptions in this theorem, the reader is referred to the original paper. The author verifies that the assumptions are satisfied in the classical cases of the operators of Legendre, Jacobi, Hermite, Laguerre, and Bessel. An extension to the case when the number of sign-changes of $L^k f(x)$ is unbounded is indicated briefly.

A. Zygmund (South Hadley, Mass.).

Mohanty, R. N. Determination of the jump of a function by its Fourier series. Bull. Calcutta Math. Soc. **34**, 99–103 (1942). [MF 7574]

This paper presents a slight contribution to the results of Szász [Duke Math. J. **4**, 401–407 (1938)] and Chow [J. London Math. Soc. **16**, 23–27 (1941); these Rev. **3**, 105]. It contains a number of misprints. It would be interesting to remove the factor $n^{-\frac{1}{2}\alpha}$ in theorem 1. *O. Szász.*

Duffin, R. J. and Eachus, J. J. Some notes on an expansion theorem of Paley and Wiener. Bull. Amer. Math. Soc. **48**, 850–855 (1942). [MF 7499]

Let $\{f_n\}$ and $\{g_n\}$ be sequences of elements of Hilbert space, $\{f_n\}$ being orthonormal. Paley and Wiener [Fourier Transforms in the Complex Domain, American Mathematical Society Colloquium Publication, vol. 19, New York, 1934, p. 100] showed that, if for some θ, $0<\theta<1$, and for every finite sequence of constants $\{a_n\}$

$$(1) \qquad \|\sum a_n(f_n-g_n)\|\leqq\theta(\sum|a_n|^2)^{\frac{1}{2}},$$

then

$$(2) \qquad (1-\theta)(\sum|a_n|^2)^{\frac{1}{2}}\leqq\|\sum a_n g_n\|\leqq(1+\theta)(\sum|a_n|^2)^{\frac{1}{2}},$$

and that $\{g_n\}$ is complete if $\{f_n\}$ is complete. The authors' results are: (A) (2) for every $\{a_n\}$ implies the existence of an orthonormal $\{f_n\}$ satisfying (1). (B) If (1) holds with strict inequality and $\theta=1$, for every (finite or infinite) $\{a_n\}$ satisfying $\sum|a_n|^2<\infty$, then $\{g_n\}$ is complete if $\{f_n\}$ is complete. (C) An analogue of (A) for $\theta=1$. (D) Let $\{C_{nk}\}$ be a matrix of constants such that $|C_{nk}|\leqq c_k$. Let $\{T_k\}$ be a sequence of bounded linear transformations with bounds $\{t_k\}$. Let $\{f_n\}$ be a complete orthonormal set. If

$$g_n=f_n+\sum_{k=1}^{\infty}C_{nk}T_k f_n,$$

and $\sum c_k t_k<1$, $\{g_n\}$ satisfies (1) and hence is complete and satisfies (2). Application: $e^{i\lambda_n x}$ is complete in $(-\pi, \pi)$ and satisfies (2) if $|\lambda_n-n|\leqq L<\pi^{-1}\log 2$ (λ_n real or complex).

E. S. Pondiczery (Princeton, N. J.).

Royall, N. N., Jr. Bounded Laplace transforms. Amer. Math. Monthly **49**, 600–604 (1942). [MF 7485]

As an analogue of the Fejér means the author associates with the Laplace transformation

$$f(s)=\int_0^\infty e^{-st}\alpha(t)dt$$

the means

$$F_R(s)=\int_0^R (1-t/R)e^{-st}\alpha(t)dt.$$

The following theorem is obtained. Let $f(s)$ converge absolutely for $\sigma>0$; then it is necessary and sufficient for $|f(s)|\leqq M$, $\sigma>0$, that $|F_R(s)|\leqq M$ for $\sigma>0$ and all $R\geqq 0$. A similar result was obtained by Stacho for Stieltjes integrals [Math. Naturw. Ber. Ungarn **33**, 20–32 (1923)] using different methods. *G. Szegö.*

Gross, B. On a new integral transformation. Anais Acad. Brasil. Ci. **12**, 317–318 (1940). (Portuguese) [MF 6578]

Gross, B. On an integral transformation which is of interest in electrotechnics. Anais Acad. Brasil. Ci. **13**, 31–50 (1941). (Portuguese) [MF 6580]

Gama, Lelio L. Note on the improper integral $\int_0^\infty B(\alpha)d\alpha/(\omega^2-\alpha^2)$.
Anais. Acad. Brasil. Ci. **13**, 51–55 (1941). (Portuguese) [MF 6581]

Gross, B. On an integral transformation which is of interest in electrotechnics. II. Anais Acad. Brasil Ci. **13**, 163–164 (1941). (Portuguese) [MF 6585]

Levi, Beppo. On the inversion of a definite integral. Publ. Inst. Mat. Univ. Nac. Litoral **3**, 117–120 (1941). (Spanish) [MF 5962]

Levi, Beppo. On an integral transformation. Publ. Inst. Mat. Univ. Nac. Litoral **3**, 121–129 (1941). (Spanish) [MF 5963]

Levi, Beppo. On an integral transformation. Anais Acad. Brasil. Ci. **13**, 185–194 (1941). (Spanish) [MF 6586]

This sequence of seven papers is concerned with the fact that under certain conditions the singular integral equation

$$(1)\qquad A(\alpha)=(2/\pi)\int_0^\infty B(\beta)(\beta/(\beta^2-\alpha^2))d\beta,\qquad \alpha>0,$$

admits the solution

$$(2)\qquad B(\beta)=(2/\pi)\int_0^\infty A(\alpha)(\beta/(\beta^2-\alpha^2))d\alpha,\qquad \beta>0,$$

where both integrals are interpreted as Cauchy principal values at their discontinuities $\beta=\alpha$ and $\alpha=\beta$, respectively.

1. The first note, by Bernardo Gross, initiates this investigation and states results which are developed in the next paper by the same author. 2. After a discussion of the formal connection of this problem with integral transformations due to Fourier, Hilbert and Hamel, the author shows that, if both $A(x)$ and $B(x)$ are piecewise smooth in $0\leqq x<\infty$ and absolutely integrable in $(0,\infty)$, then the relation (1) implies (2) and conversely. Also if $B(\beta)=\int_0^\infty\phi(\alpha)\cos(\alpha\beta)d\alpha$, where $\int_0^\infty|\phi(\alpha)|d\alpha<\infty$, then $A(\alpha)$ defined by (1) is the Fourier sine transform of the same function: $A(\alpha)=\int_0^\infty\phi(\beta)\sin(\alpha\beta)d\beta$. Similar results are established if $B(\beta)$ is periodic and admits of an absolutely convergent cosine Fourier expansion. 3. The third paper, by Lelio L. Gama, is concerned with sufficient conditions on $B(\alpha)$ which will insure the existence of the integral in the title as an improper integral at $\alpha=0$, $\alpha=\infty$ and as a principal value at $\alpha=\omega$. 4. In the fourth paper B. Gross assumes that

$$(3)\quad A(\alpha)=\int_0^\infty\phi(\sigma)\sin(\sigma\alpha)d\sigma,\quad B(\beta)=\int_0^\infty\phi(\sigma)\cos(\sigma\beta)d\sigma,$$

where $\int_0^\infty|\phi(\sigma)|d\sigma<\infty$. In view of his results of the second paper, these assumptions imply the relations (1), (2). Assuming now that $\int_0^\infty|\sigma\phi(\sigma)|d\sigma<\infty$, the integrals (3) may be formally differentiated and express $A'(\alpha)$, $B'(\beta)$ as cosine and sine transforms of $\sigma\phi(\sigma)$ and $-\sigma\phi(\sigma)$, respectively. This now again implies that $A'(\alpha)$, $B'(\beta)$ are related by a set of equations analogous to (1), (2).

5. In his first paper Beppo Levi announces results of the following two papers, which are identical. He investigates the problem more directly and arrives at the following much more general theorem. Let the real function $A(\alpha)$ be bounded and continuous in the range $0<\alpha<\infty$ such that $\int^\infty|A(\alpha)|\alpha^{-1}d\alpha$ converges. Then equation (1) admits the solution $B(\beta)$ defined by (2). *I. J. Schoenberg.*

Rao, D. V. B. Two inversion formulae. Bull. Calcutta Math. Soc. **34**, 79–85 (1942). [MF 7532]

The formulae are

$$\int_a^\infty dx\int_{-\infty}^\infty\left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right\}f(t)dt=2f(s)$$

(ν an integer) [Fox, Proc. London Math. Soc.(2) **26**, 395–402 (1927)] and a more elaborate one involving Bessel functions of six orders [Mohan, Proc. Benares Math. Soc. **15**, 1–20 (1933)]. Their validity was known for a class of Fourier transforms $f(s)$. They are here shown to apply if $f(s)$ is defined by

$$f(s)=\int_a^b\frac{J_\beta(s-u)}{(s-u)^\beta}\varphi(u)du,$$

where $-1<a<b<1$, $\beta>-\frac{1}{2}$, and $|\varphi(u)|$ is integrable, or by other similar formulae. *E. S. Pondiczery.*

Guinand, A. P. Summation formulae and self-reciprocal functions. III. Quart. J. Math., Oxford Ser. **13**, 30–39 (1942). [MF 7447]

[The two previous parts appeared in Quart. J. Math., Oxford Ser. **9**, 53–67 (1938) and **10**, 104–118 (1939)].

The author proves that the function

$$F(x)=x^{-1}\{\sum_{n\leqq x}{}'\Lambda(n)-x\}+\tfrac{1}{2}\sum_\rho(x^{\rho-1}/\rho)+A(x),$$

where

$$A(x)=\begin{cases}(4x)^{-1}\log((1+x)/(1-x)) & \text{for } x<1,\\ (2x)^{-1}\log(2\pi+j)+(4x)^{-1}\log(x^2-1) & \text{for } x>1,\end{cases}$$

satisfies the relation

$$F(x)=x^{-1}F(x^{-1})-x^{-1}\int_{x^{-1}}^\infty F(t)dt/t.$$

This leads to a general summation formula which the author does not substantiate in detail. He also shows that, for $0<\Re(\alpha)<\frac{1}{2}$, a function

$$F_\alpha(x)=x^{\alpha-1}\{\sum_{n\leqq x}{}'\Lambda(n)n^{-\alpha}-x^{1-\alpha}/(1-\alpha)\}+\tfrac{1}{2}\sum_\rho(x^{\rho-1}/(\rho-\alpha))+A_\alpha(x)$$

satisfies the relation

$$F_\alpha(x)=x^{-1}F_\alpha(x^{-1})+(2\alpha-1)x^{\alpha-1}\int_{x-1}^{\infty}t^{-\alpha-1}F_\alpha(t)dt.$$

However, this case presupposes the Riemann hypothesis.

S. Bochner (Princeton, N. J.).

Mital, P. C. Operational images of self-reciprocal functions. J. Indian Math. Soc. (N.S.) **6**, 25–32 (1942). [MF 7232]

If $f(x)$ is self reciprocal as a Hankel transform of order ν it may be represented as a Mellin integral with integrand involving x through the term x^{-s} alone, $0<\Re(s)<1$ [Hardy and Titchmarsh, Quart. J. Math., Oxford Ser. 1, 196–231 (1930)]. The Laplace transform of $f(x)$ therefore amounts to replacing x^{-s} by $\int_0^\infty e^{-px}x^{-s}dx=p^s\Gamma(1-s)$, if the order of integration may be interchanged [but the author gives no justification for this interchange]. The formal result of replacing x^{-s} by $p^s\Gamma(1-s)$, and an analogous replacement when, instead of the Hankel transform, one has the sine transform, constitutes the content of the writer's two theorems.

D. G. Bourgin (Urbana, Ill.).

Differential Equations

Sedgwick, W. F. On the theory of successive radioactive transformations. Proc. Cambridge Philos. Soc. **38**, 280–289 (1942). [MF 7811]

Sedgwick, W. F. Note on a paper "On the theory of successive radioactive transformations." Proc. Cambridge Philos. Soc. **39**, 68 (1943). [MF 7970]

In mathematical terms the author is concerned with the explicit solution of a finite system of differential equations $dn_k/dt=-\lambda_k n_k+\lambda_{k-1}n_{k-1}$, where $\lambda_k>0$ are constants and $\lambda_0=0$. The same solution occurs in a paper by Feller [Acta Bioth. Ser. A. **5**, 11–40 (1939); these Rev. **1**, 22; Trans. Amer. Math. Soc. **48**, 488–515 (1940), example (i); these Rev. *2*, 101] and N. Arley [Mat. Tidsskr. B **1939**, 49–51; these Rev. **1**, 72].

W. Feller (Providence, R. I.).

González, Mario O. Relations between the classical methods of solving ordinary differential equations. Revista Soc. Cubana Ci. Fís. Mat. **1**, 4–12 (1942). (Spanish) [MF 7191]

For a first order differential equation (I): $M(x, y)dx+N(x, y)dy=0$ the author presents a specific formula for an integrating factor $\lambda(x, y)$ in terms of a set of canonical variables $u=u(x, y)$, $v=v(x, y)$ with $\partial(u, v)/\partial(x, y)\neq 0$. Conversely, he shows that, if an integrating factor of (I) is known, then corresponding canonical variables for this equation can be determined by quadratures. Finally, the formula for an integrating factor of (I) is generalized to give a Jacobi multiplier for a linear homogeneous partial differential equation of the first order.

W. T. Reid.

Azevedo do Amaral, Ignacio M. On the integration of ordinary linear differential equations and the solution of integral equations of the first kind. Anais Acad. Brasil. Ci. **14**, 87–97 (1942). (Portuguese) [MF 7260]

The author considers the differential equation

$$\sum_{i=0}^{n}X_i y^{(n-i)}=F(x),$$

where the coefficients X_i are functions of the independent variable x. Particular solutions of this equation are obtained in the form

$$(*)\qquad y=\int_a^b\frac{K(x,\alpha)d\alpha}{\int_a^b\theta(x,\alpha)dx}\int_a^b F(x)\frac{P(x,\alpha)dx}{\int_a^b P(x,\alpha)d\alpha},$$

where $K(x, \alpha)$, $\theta(x, \alpha)$ are suitably determined functions of x and α, and $P(x, \alpha)$ is arbitrarily chosen. [This is the case $p=1$ of the paper reviewed below.] The author indicates some applications of his results to the solution of certain integral equations of the Fredholm and Volterra types.

M. A. Basoco (Lincoln, Neb.).

Azevedo do Amaral, Ignacio M. On the integration of linear partial differential equations. Anais Acad. Brasil. Ci. **14**, 83–86 (1942). (Portuguese) [MF 7259]

The author considers the partial differential equation of order n in p independent variables

$$\sum X_{i_1,i_2,\cdots,i_p}\partial^{i_1+i_2+\cdots+i_p}/\partial^{i_1}x_1\partial^{i_2}x_2\cdots\partial^{i_p}x_p=F(x_1, x_2, \cdots, x_p),$$

where the X's are functions of x_k, $k=1, 2, \cdots, p$. It is shown that, subject to certain conditions imposed by the method used, a particular solution can be obtained in a form similar to that of (*) in the preceding review; x now stands for a point $(x_1, \cdots, x_p)$ and the integrals with respect to x stand for p-tuple integrals. Reference is made to a previous note by the author [Rend. Acad. Napoli; no date is given].

M. A. Basoco (Lincoln, Neb.).

Newell, Homer E., Jr. The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain. Duke Math. J. **9**, 245–258 (1942). [MF 6865]

On représente par le symbole matriceal

$$(d/dx)Y(x, \lambda)=\{\lambda R(x)+Q(x, \lambda)\}Y(x, \lambda),$$

où $R(x)=\{\delta_{ij}r_j(x)\}$, $Q(x, \lambda)=\{q_{ij}(x, \lambda)\}$, un système d'équations différentielles linéaires pour le système de fonctions inconnues $Y(x, \lambda)=\{y_i(x, \lambda)\}$. On suppose que les fonctions $r_j(x)$ et $q_{ij}(x, \lambda)$ sont analytiques par rapport à x et que les $q_{ij}(x, \lambda)$ admettent une représentation asymptotique pour $|\lambda|\to\infty$

$$q_{ij}(x, \lambda)\sim\sum_{\nu=0}^{\infty}\lambda^{-\nu}q_{ij}^{(\nu)}(x)\quad\text{avec } q_{ij}^{(0)}(x)\equiv 0.$$

Il s'agit de déterminer des conditions suffisantes pour que le système admette une solution tendant asymptotiquement à $E(x, \lambda)=\{\delta_{ij}\exp\lambda\int^x r_j(x)dx\}$. Dans un memoire précédent R. E. Langer [Trans. Amer. Math. Soc. **46**, 151–190 et correction 467 (1939)] avait considéré le cas dans lequel les fonctions r_j et q_{ij} sont analytiques et bornées par rapport à x et les différences $r_i(x)-r_j(x)$ $(i\neq j)$ ne s'annulent pas dans la région considérée. L'auteur démontre que la solution du système peut se représenter dans la forme $Y(x, \lambda)=P(x, \lambda)E(x, \lambda)$, où $P(x, \lambda)$ est une matrice analytique en x et tendant uniformément à la matrice unité quand $\lambda\to\infty$, dans le cas que les coefficients possèdent des poles sur le contour de la région de définition. Précisément, on suppose que, x_0 pouvant être un pole pour les fonctions $r_j(x)$ et $q_{ij}(x, \lambda)$, il existe un secteur de sommet x_0 à l'intérieur duquel ces fonctions sont régulières et les différences $r_i(x)-r_j(x)$ ne s'annulent pas, pouvant cependant avoir comme zéro le point x_0.

B. Levi (Rosario).

Reid, William T. A new class of self-adjoint boundary value problems. Trans. Amer. Math. Soc. **52**, 381–425 (1942). [MF 7454]

The writer considers differential systems of the form

$$(1)\qquad \begin{cases} y_i' = \sum_{j=1}^{n} [A_{ij}(x)+\lambda B_{ij}(x)]y_j, \\ s_i[y] \equiv \sum_{j=1}^{n} [M_{ij}y_j(a)+N_{ij}y_j(b)] = 0, \quad i=1, \cdots, n. \end{cases}$$

In vector and matrix notation, these may be written

$$L[y]=\lambda B(x)y, \quad s[y]\equiv My(a)+Ny(b)=0,$$
$$L[y]\equiv y'-A(x)y.$$

The adjoint system is

$$(2)\qquad M[z]=-\lambda zB(x), \quad s[z]=z(a)P+z(b)Q=0,$$
$$M[z]\equiv z'+zA(x),$$

the matrices P and Q being connected with M and N in a certain way. The system (1) is said to be self-adjoint under $z=T(x)\cdot y$ (T nonsingular, of class C') if, for arbitrary values of λ, y satisfies (1) if and only if $z=Ty$ satisfies (2). If y is a solution of (1), then

$$(3)\qquad \int_a^b y\tilde{T}L[y]dx=\lambda\int_a^b ySydx, \qquad S=\tilde{T}B,$$

$\tilde{T}$ being the transpose of T. Bliss [Trans. Amer. Math. Soc. **44**, 413–428 (1938)] has said that a system (1) is definitely self-adjoint if (i) it is self-adjoint under some $T(x)$, (ii) the matrix S in (3) is symmetric, (iii) the quadratic form uSu is positive semidefinite for each x on (a, b) and (iv) there exists no nonidentically vanishing solution of $L[y]=0$, $s[y]=0$ for which $B(x)y(x)\equiv 0$ on (a, b). The writer defines a system (1) to be H-definitely self-adjoint if the quadratic functional $H[y]\equiv\int_a^b y\tilde{T}L[y]dx$ is positive for all vectors y of class C' for which $s[y]=0$, $B(x)y(x)\not\equiv 0$, and there exists a continuous $g(x)$ such that $L[y]=Bg$. Many results concerning characteristic values and their multiplicities, indices and extremum properties and many other results are proved for such systems. Further results on Bliss's definitely self-adjoint systems and certain other systems are obtained with the aid of the writer's new theory. Finally an important system connected with the problem of Bolza in the calculus of variations is shown to be H-definitely self-adjoint. *C. B. Morrey, Jr.*

Whyburn, William M. Differential equations with general boundary conditions. Bull. Amer. Math. Soc. **48**, 692–704 (1942). [MF 7267]

This is an address given before the American Mathematical Society. It consists in the main of a review of boundary value problems for ordinary nonsingular differential equations (in the real domain), with conditions at one or more points in the interval of definition. In particular, a discussion of integral boundary conditions is given, a field connecting naturally with integral equations. The elements of novelty in this address are mostly in the investigation of the types of boundary conditions. *W. J. Trjitzinsky.*

Titchmarsh, E. C. An eigenfunction problem occurring in quantum mechanics. Quart. J. Math., Oxford Ser. **13**, 1–10 (1942). [MF 7444]

This paper deals with the Dirac first order relativistic wave equation of the hydrogen atom, which has the form (1) $L\psi=w_n\psi$, where ψ is a two-vector with components $\psi_1(x)$ and $\psi_2(x)$. The eigenvalues w_n were found by Gordon [*Z. Phys.* **48**, 11–14 (1928)]. The author here derives the expansion of an "arbitrary" two-vector ψ in terms of the eigenfunctions of (1), using the method he has recently developed for the case of Sturm-Liouville expansions and their generalizations [Quart. J. Math., Oxford Ser. **11**, 129–145 (1940); ibid. **12**, 33–50, 89–107, 154–166 (1941); these Rev. **2**, 53; **3**, 121, 235]. The expansion is obtained by applying the calculus of residues to the representation by means of the Fourier integral of the solution of a partial differential equation related to (1). The final formulas are given in terms of confluent hypergeometric functions. There is some discussion of conditions that ψ must satisfy in order that the method be valid. It is not known whether the results have any physical application. *O. Frink.*

Muggli, H. Differentialgleichungen unendlich hoher Ordnung. Comment. Math. Helv. **14**, 381–393 (1942).

Continuation of the work begun in the author's thesis [Comment. Math. Helv. **11**, 151–179 (1938)]. It is supposed that $L(z)$ is an entire function of order $\rho<1$ and that a ray $\arg z=\Phi$ exists having a positive distance from the zeros of $L(-z)$. Further, $g(z)=\sum_0^\infty a_n z^{-n-1}$ is a given function, the Borel transform of $G(w)=\sum_0^\infty a_n w^n/n!$, $\Im$ the conjugate diagram (in the sense of Pólya) of $g(z)$, $k(\varphi)$ its function of support, so that $k(-\varphi)$ is the indicator of $G(w)$, that is,

$$\limsup_{R\to\infty} (1/R)\log|G(Re^{i\varphi})| = k(-\varphi).$$

Then the integral

$$f(z)=\int_0^{e^{i\Phi}\infty} (e^{-\zeta z}G(\zeta)/L(-\zeta))d\zeta,$$

taken along the ray $\arg\zeta=\Phi$, converges in the half-plane $\Re(ze^{i\Phi})>k(-\Phi)$ and satisfies there the equation $L(d/dz)f(z)=g(z)$. If $L(z)$ has only a finite number of zeros in a sector, the solution can be continued analytically by varying the path of integration. The author shows by an example that the domain of existence of $f(z)$ may reduce to a sector of opening 3π on the Riemann surface of the logarithm.

E. Hille (New Haven, Conn.).

Aronszajn, N. Le correspondant topologique de l'unicité dans la théorie des équations différentielles. Ann. of Math. (2) **43**, 730–738 (1942). [MF 7403]

G. D. Birkhoff and Kellogg [Trans. Amer. Math. Soc. **23**, 96–115 (1922)] proved fundamental existence theorems for differential equations by means of a theorem on fixed points of a transformation in a Banach space. The present paper studies the set of fixed points further and obtains a topological description of the family of solutions of the differential equations satisfying a given set of initial equations. The family of solutions arises because the differential equations are not assumed to have enough regularity (for example, Lipschitz conditions) to ensure a unique solution; however, it is assumed that the differential equations can be approximated by ones admitting a unique solution. With a natural topology in the space of solutions, that space is shown to be homeomorphic to the limit of a decreasing sequence of absolute retracts [see K. Borsuk, Fund. Math. **17**, 152–170 (1931)]. As an application a theorem of Kneser is proved [see M. Müller, Math. Z. **28**, 349–355 (1928)]. *W. Kaplan* (Ann Arbor, Mich.).

Aronszajn, Nathan and Weinstein, Alexander. On the unified theory of eigenvalues of plates and membranes. Amer. J. Math. **64**, 623–645 (1942). [MF 7167]

The present paper provides detailed proofs of the remarkable facts described in a previous note [Proc. Nat. Acad. Sci. U. S. A. **27**, 188–191 (1941); these Rev. **3**, 44]. It is primarily concerned with the eigenvalue problem for the clamped plate, that is, the eigenvalue problem for the differential equation $\Delta\Delta w-\lambda w=0$ for a domain S with boundary C in the x,y plane under the boundary conditions (a) $w=0$, (b) vanishing derivatives at C. This problem is replaced by the variational problem of minimizing the quotient $I(w)/H(w)$ with

$$I(w)=\int\int_S(\Delta w)^2dxdy,\quad H(w)=\int\int_S w^2dxdy,$$

where w is orthogonal to the solutions of $n-1$ preceding problems and furthermore satisfies the boundary conditions (a) and (b) and obvious conditions of continuity. If the condition (b) is dropped, the much simpler problem of the supported plate arises, which is equivalent to that of the clamped membrane. On the other hand, as anticipated by Zaremba in 1909, condition (b) can be replaced by the orthogonality condition $H(\Delta w, p)=0$ for all regular harmonic functions p in S with $H(p)<\infty$. The essential new point is now to introduce a sequence of intermediate problems: A complete system $p_1, p_2, p_3, \cdots$ of harmonic functions in S is considered and condition (b) is replaced by $H(\Delta w, p_i)=0$ for $i=1, \cdots, m$. The minimum values of this new problem are easily seen to be not greater than the corresponding values for the clamped plate, for which they thus provide lower bounds. The authors give an analysis of these intermediate problems, an existence proof and, above all, the proof that, as m tends to ∞, the solution of the intermediate problem tends to that of the clamped plate. For the proof the variational problems are transformed by means of Green's function into a form in which no derivatives appear. Use is made of results by others concerning existence theorems for the problem of the clamped plate. The paper contains, furthermore, an analysis of the relation between the variational problems and the corresponding eigenvalue problems of the differential equation. Extensions to the case of partly clamped, partly supported plates and to the problem of buckling are briefly indicated. The theory is shown to be valid under rather liberal assumptions on the nature of the domain S.

R. Courant (New York, N. Y.).

Badesco, Radu. Sur un procédé mixte pour résoudre les problèmes de Cauchy et de Goursat relatifs à l'équation des télégraphistes. Enseignement Math. **38**, 231–241 (1942). [MF 7300]

Consider the telegrapher's equation (1) $V_{xx}-V_{tt}+V=0$. Suppose (2) $V(x,t)=\sum t^nR_n(x)/n!$ is a solution of (1). Then (3) $R_n(x)=R_{n-2}(x)+(d^2/dx^2)R_{n-2}(x)$, by formal substitution in (1). If one writes D for the operator $1+d^2/dx^2$, then the author's deductions amount to (4) $R_{2n+i}(x)=D^nR_i$, $i=0, 1$. Let K_0 be the Fourier cosine transform of R_0 and K_1 the Fourier sine transform of R_1. From (4) it is obvious that the corresponding transforms of R_{2n}, R_{2n+1} are $(1-u^2)^nK_0$ and $(1-u^2)^nK_1$, respectively. Now from the developments of $\cos t(u^2-1)^{\frac{1}{2}}$ and of $\sin t(u^2-1)^{\frac{1}{2}}$ one gets

$$(5)\quad V(x,t)=\int_0^\infty (K_0(u)\cos t(u^2-1)^{\frac{1}{2}}\cos xu + K_1(u)\sin t(u^2-1)^{\frac{1}{2}}\sin xu)du.$$

If Cauchy data, vanishing outside a finite interval of the x axis, are given, then K_0 and K_1 may be determined. The author asserts that summability of R_i, K_i, $K_1(u^2-1)^{-\frac{1}{2}}$, $i=0, 2$, is sufficient for V, defined by (5), to be a solution of the Cauchy problem. These conditions seem inadequate to the reviewer since they do not imply any differentiability properties of $V(x,t)$. The point of the paper is the heuristic determination of (5) starting from (2), after which (5) may be independently shown to be a solution in cases where (2) is not valid. An analogous development is given for the case where the data are prescribed along the characteristics.

D. G. Bourgin (Urbana, Ill.).

Wilson, James T. Surface waves in a heterogeneous medium. Bull. Seismol. Soc. America **32**, 297–304 (1942). [MF 7478]

This paper is concerned with the solution of two boundary value problems. (I) (Love waves)

$$(*)\qquad \mu\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial z^2}\right)+\frac{\partial\mu}{\partial z}\frac{\partial v}{\partial z}=\rho\frac{\partial^2v}{\partial t^2},$$

where v is the y displacement in a semi-infinite medium, μ the rigidity and ρ the density. (*) is solved under the assumptions (i) $\mu=u_0e^{cz}$, (ii) $(\mu/\rho)^{\frac{1}{2}}=b_0e^{\beta z}$, (iii) $v=V(z)e^{ik(x-ct)}$, (iv) $\partial V/\partial z=0$, $z=0$. The particular forms chosen for μ and ρ enable the author to reduce (*) to an equation which may in turn be reduced to a form of a Bessel equation. (II) A similar mathematical problem is encountered for Rayleigh waves if one assumes a constant density and a rigidity variation $(\mu/\rho)^{\frac{1}{2}}=b_0e^{\beta z}$. This problem involves the solution of a system of two partial differential equations involving x and z displacements under given boundary conditions.

A. E. Heins (Cambridge, Mass.).

Leibenson, L. Die Bewegung einer gashaltigen Flüssigkeit in einem porösen Mittel. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 411–422 (1941). (Russian. German summary) [MF 7324]

The problem mentioned in the title leads, in the usual way, to an equation of diffusion

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2=M(p)\partial\phi/\partial t,$$

where ϕ is the potential of the motion and p the pressure. The problem is of interest in geophysics and the oil industry.

S. Bergman (Providence, R. I.).

Loomis, L. H. and Widder, D. V. The Poisson integral representation of functions which are positive and harmonic in a half-plane. Duke Math. J. **9**, 643–645 (1942). [MF 7339]

In this note the authors prove that a necessary and sufficient condition that a function be harmonic and positive in a half plane is that it be representable as a Poisson integral with positive integrator function. Earlier proofs have depended on the corresponding theorem for a circle and on the conformal mapping of the circle on the half plane. The proof depends on a succession of three preliminary theorems: (1) If $\varphi(x)/(1+x^2)\varepsilon L$ in $(-\infty, \infty)$ and is continuous at x_0, then

$$F(x,y)=\pi^{-1}\int_{-\infty}^{\infty}y\varphi(t)dt/[y^2+(t-x)^2]$$

is harmonic in $y>0$ and $F(x_0, 0+)=\varphi(x_0)$. (2) If $f(x,y)$ is harmonic and nonnegative for $y>0$, then $f(x,y)/(1+x^2)\varepsilon L$

in $(-\infty, \infty)$ for every $y>0$. (3) If $f(x, y)$ is harmonic and nonnegative for $y \geqq 0$, vanishing for $y=0$, then $f(x, y)=ky$, where $k>0$. If $f(x, y)$ is positive and harmonic in the upper half plane and

$$F_\delta(x, y)=\pi^{-1}\int_{-\infty}^{\infty} yf(t, \delta)dt/[y^2+(t-x)^2],$$

then it is shown that, as $\delta \to 0$, $F_\delta \to f+\text{const. } y$, and the representation of f is obtained by the use of the Helly-Bray thereom. *J. W. Green* (Rochester, N. Y.).

Malkin, N. Conditions for use of Stokes' formula in determining the earth figure from observations on gravity. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 8–10 (1942). [MF 7609]

Zamorev, A. Untersuchung der zweidimensionalen inversen Aufgabe der Potentialtheorie. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 487–500 (1941). (Russian. German summary) [MF 7329]

We denote by $\varphi(x, y)$ the two dimensional potential of a mass lying in the upper half-plane; φ_x and φ_y are its two derivatives and φ_y is always given along the x-axis. If γ is the density of the mass and

$$\int\int \gamma(x+iy)^n dxdy=a_n+ib_n$$

its moments, then it is shown that

$$\int\int \gamma e^{-\alpha y}e^{\alpha iz}dxdy=(1/2\pi k^2)\int_{-\infty}^{\infty}\varphi_y(x, 0)e^{\alpha iz}dx$$
$$=\sum_{n=0}^{\infty}((i^n\alpha^n)/n!)(a_n+ib_n),$$

where k denotes a constant. Hence it follows that the moments are determined if we know φ_y on the x-axis. Different infinite integrals are deduced which, if convergent, furnish the extension of φ_x and φ_y into the lower and upper half-plane. A homogeneous body is supposed to be bounded in $a \leqq x \leqq b$ by two analytic curves $y=u(x)$, $y=v(x)$ for which $u(a)=u(b)=v(a)=v(b)=h$ and $u(z)$, $v(z)$ can be extended as functions of a complex variable into a neighborhood of the segment $a \leqq x \leqq b$, $y=h$; if z is on the boundary of this neighborhood and x on the segment then $|u(z)/(z-x)|<1$. Under these conditions the expression $-\varphi_x+i\varphi_y$ of the gravitational field is found in the form of an integral along the segment of an analytic function and shown to have singular points only at the endpoints of the segment. Finally an ingenious solution of the inverse problem is given: φ_y along the x-axis, γ, (a, b) and $v(z)$ are supposed to be known, the latter satisfying the above conditions. An integral expression for $u(x)$ is given which determines completely the body which causes the given gravitational field. *František Wolf* (Berkeley, Calif.).

Zamorev, A. A. Determination of the form of a body by the derivatives of the external gravitational potential. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1942**, 48–54 (1942). (Russian. English summary) [MF 7565]

The author treats the same subject as in the paper reviewed above in a slightly more detailed and polished form, methodically relying exclusively on the theory of analytic continuation. *František Wolf* (Berkeley, Calif.).

Theory of Probability

***Holmes, Maurice C. An Outline of Probability and Its Uses.** Burgess Publishing Co., Minneapolis, Minn., 1936. viii+119 pp. $1.50.

This book might serve as a syllabus and problem list for an introductory course in applied probability. It would have to be supplemented in a thoroughgoing way, however, as to the mathematical discussion and the fundamental ideas, in both which respects it is quite unsatisfactory. The book presents nothing new from either a scientific or a pedagogical point of view. *B. O. Koopman.*

Chung, Kai-Lai. On mutually favorable events. Ann. Math. Statistics **13**, 338–349 (1942). [MF 7244]

Two events are said to be mutually favorable provided the probability of one is increased when the other is known to have occurred. This concept is extended to the case of n events in a manner analogous to the corresponding extension of the concept of independence. The author establishes an inequality with respect to the probabilities of mutually favorable events. He then introduces a stronger concept of favorableness in terms of certain disjunctions and derives an analogous inequality. As a special case of this latter inequality he obtains a known result in number theory. *A. H. Copeland* (Ann Arbor, Mich.).

Battin, I. L. On the problem of multiple matching. Ann. Math. Statistics **13**, 294–305 (1942). [MF 7239]

This paper contains a generalization of the matching problem to the case of three or more decks of cards. The author constructs a suitable generating function by means of which the various probabilities and the moments of the various distributions can be computed. This function is a generalization of one introduced by Wilks to study the problem of the matching of two decks. *A. H. Copeland* (Ann Arbor, Mich.).

Gontcharoff, W. Sur la distribution des cycles dans les permutations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 267–269 (1942). [MF 7622]

Consider a permutation $a_1, a_2, \cdots, a_n$ of the elements $1, 2, \cdots, n$. The author proves, among others, the following theorems: (1) If m denotes the number of cycles of the permutation, then

$$\lim \text{Prob.}\left\{\alpha<\frac{m-\log n}{\sqrt{2\log n}}<\beta\right\}=\pi^{-\frac{1}{2}}\int_\alpha^\beta e^{-t^2}dt.$$

(2) The mathematical expectation of the mean length of the cycles, i.e., of n/m, is asymptotically $n/\log n$. (3) Denote by $p_m^{(n)}(r)$ the probability that the number of cycles having r terms is m, then

$$\lim_{n\to\infty} p_m^{(n)}(r)=\frac{1}{m!}\left(\frac{1}{r}\right)^m e^{-1/r}.$$

P. Erdös (Philadelphia, Pa.).

Gnedenko, B. On locally stable probability distributions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 263–266 (1942). [MF 7621]

Let ξ be an infinitely divisible random variable; then for any n one has $\xi=\xi_{n1}+\xi_{n2}+\cdots+\xi_{nn}$, where the ξ_{nk} are mutually independent and have the same distribution function. The author states (without proof) a series of new

results on the limit distribution to which may converge the distributions of the variables $\eta_{nk}=(\xi_{nk}-b_n)/a_n$, where a_n and b_n are constants. In particular: The class of limit distributions of η_{nk} coincides with that of stable distributions. The variable ξ is called locally stable (locally Gaussian) if the distribution functions of η_{nk} tend to a limit distribution (tend to a Gaussian distribution) as $n\to\infty$. The author gives n.a.s. conditions for ξ to be locally stable or Gaussian. If one chooses, in particular, $a_n=an^{-\frac{1}{2}}$, then η_{nk} are Gaussian distributed in the limit if, and only if, ξ contains a Gaussian component. Finally: let $\xi(t)$ be a homogeneous stochastic process with independent increments. Then $\xi(t)$ is infinitely divisible. Now in order that it be possible to choose constants a_t and b_t in such a way that the distribution function of $(\xi(t)-\xi(0))/a_t-b_t$ tends to a limit distribution as $t\to\infty$ it is n.a.s. that the $\xi(t)$ be locally stable.

W. Feller (Providence, R. I.).

Gnedenko, B. Investigation of the growth of homogeneous random processes with independent increments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 3–4 (1942). [MF 7623]

Let the random variable $\xi(t)$ stand for a stochastic process with independent increments (Markoff type) which is homogeneous both in time and space. According to Khintchine a non-decreasing function $u(t)$ is called an upper bound of the process if for a conveniently chosen constant γ the equality

$$\limsup_{t\to\infty}\frac{|\xi(t)-\xi(0)-\gamma t|}{u(t)}=0$$

holds with probability 1. The author states without proof: (i) the function $(t\log\log t)^{\frac{1}{2}}$ is never an upper bound; (ii) if $\omega(t)\to\infty$ then $\omega(t)(t\log\log t)^{\frac{1}{2}}$ is an upper bound provided the dispersion of $\xi(t)$ is finite; (iii) if the dispersion is infinite, a function $\omega(t)$ can be found such that $\omega(t)\to\infty$ and nevertheless $\omega(t)(t\log\log t)^{\frac{1}{2}}$ is not an upper bound; (iv) if, and only if, the dispersion is finite, we have

$$\limsup_{t\to\infty}\frac{|\xi(t)-\xi(0)-\gamma t|}{(t\log\log t)^{\frac{1}{2}}}=a,$$

where a is a constant. *W. Feller* (Providence, R. I.).

Doob, J. L. What is a stochastic process? Amer. Math. Monthly **49**, 648–653 (1942). [MF 7814]

An expository article in which the author introduces the fundamental notion of a stochastic process and discusses its relation to measure theory.

Bouchman, E. N. Problems of congestion in telephony. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 247–256 (1942). (Russian. English summary) [MF 7755]

The application of theoretical solutions in telephony suffers from a number of defects. Many valuable works are either not used at all or wrongly applied because the results are interpreted erroneously. As an example, we may point out the confusion of number of calls and traffic actually carried, in the widely used formulae of Erlang and Fry (for the probability of loss-calls). Many erroneous theories have received wide application in practice. The present paper points out the changes which must be made in the mathematically exact formula of Khintchine for its application in actual practice. The article also points out several means of application of the methods of the theory of probability in related fields: in particular, in telegraphy, for example, in calculations of traffic capacity and in calculations of the number of messenger boys necessary.

From the author's summary.

Jeffreys, Harold. Probability and quantum theory. Philos. Mag. (7) **33**, 815–831 (1942). [MF 7553]

Theoretical Statistics

Kenney, J. F. Characteristic functions in statistics. Nat. Math. Mag. **17**, 51–67, 99–114 (1942). [MF 7466]

This is an elementary exposition of fundamental properties of characteristic and generating functions from the point of view of a practical statistician. *M. Kac.*

Gabriel, Fritz. Graphisch-rechnerisches Verfahren zur schnellen Ermittlung von Trefferprozenten unter einfachen und erschwerten Bedingungen. "Trefferspinne." Luftfahrtforschung **19**, 231–235 (1942). [MF 7439]

Given a chart for the two-dimensional Gaussian with radial symmetry, that is to say, a division of the plane by circles and radii into areas of equal probability, the author indicates a simple method for deducing the corresponding values for an elliptic distribution, that is, a Gaussian with vanishing correlation coefficient. *W. Feller.*

Evans, W. Duane. The standard error of percentiles. J. Amer. Statist. Assoc. **37**, 367–376 (1942). [MF 7144]

An approximate formula is derived for the standard deviation of the estimate of any given percentile based on a sufficiently large sample. The method is developed for the general case of a stratified sample, and requires no assumptions on the distribution of the parent population.

Z. W. Birnbaum (Seattle, Wash.).

Lévy, Paul. Sur la détermination expérimentale de la loi des erreurs. Enseignement Math. **38**, 227–231 (1942). [MF 7299]

The author studies various possibilities for testing the hypothesis that a random sample is derived from a normal population with given parameters, the only alternative hypothesis being that the probability law is given by $\frac{1}{2}e^{-|x|}$. Numerical calculations show that, with a sample size of 500, the decision is by no means safe. *W. Feller.*

Bose, Purnendu. On the exact distribution of the ratio of two means belonging to samples drawn from a given correlated bivariate normal population. Bull. Calcutta Math. Soc. **34**, 139–141 (1942). [MF 7579]

Fieller [Biometrika **24**, 382–403 (1932)] derived by rather direct integration in the $\bar{x},\bar{y}$ plane the sampling distribution of $\bar{y}/\bar{x}$, where $\bar{y}$ and $\bar{x}$ are means of y and x in a sample from a normal bivariate distribution. In the present paper the author arrives at the same result by a slightly different method. *S. S. Wilks* (Princeton, N. J.).

Kibble, W. F. A two-variate gamma type distribution. Sankhyā **5**, 137–150 (1941). [MF 7349]

"In this paper a distribution function bilinear in Laguerre polynomials is found for two variates each of which has a "type III," or gamma type, distribution. In the special case in which the two variates have similar distributions (that

is, have the same coefficient of variation), the distribution function can also be expressed in terms of a modified Bessel function. The approach to the problem is as follows: the variance, in a sample of given size from a normally distributed population, has a distribution of type III; we investigate the distribution of the variances of two variates in samples from a normal two variate distribution . . . an extension is made to a two-variate distribution of normal type in one variate and gamma type in the other."

The author, starting with the two-variate normal correlated distribution in X and Y, makes the substitution $x=X^2/2$, $y=Y^2/2$, and obtains the distribution $\phi(x, y)dxdy$ with the moment generating function (m.g.f.)

$$G(\alpha, \beta)=[(1-\alpha)(1-\beta)-\alpha\beta\rho^2]^{-\frac{1}{2}}.$$

If, now, $-\frac{1}{2}$ is replaced by $-n/2$, there is obtained the m.g.f. for the variances in a sample of n. Then, setting $-n/2=-p$, the author expands $G(\alpha, \beta)(1-\alpha)^p(1-\beta)^p$ by the binomial theorem, introduces Laguerre polynomials, and obtains

$$\phi(x, y)=1+\sum_1^\infty \rho^r L_r(x, p)L_r(y, p)/[r!p(p+1)\cdots(p+r-1)].$$

E. L. Dodd (Austin, Tex.).

Banerjee, D. P. Note on the limit of correlation and regression coefficients in mingled records. Math. Student 9, 155–157 (1941). [MF 7031]

Let x and y be two variates and let $S_1, \cdots, S_k$ be k samples drawn from the bivariate population (x, y). Furthermore let S be the sample we obtain by pooling the samples $S_1, \cdots, S_k$. The author derives some limits for the correlation and regression coefficients in the sample S. These limits are expressed in terms of certain statistics calculated from the samples $S_1, \cdots, S_k$. *A. Wald.*

Gebelein, Hans. Das statistische Problem der Korrelation als Variations- und Eigenwertproblem und sein Zusammenhang mit der Ausgleichsrechnung. Z. Angew. Math. Mech. 21, 364–379 (1941). [MF 7656]

Let $w(x, y)$ be a bivariate frequency function. The purpose of the paper is to define a new measure of correlation, K^2, which vanishes if, and only if, the two variates are independent (that is to say, if $w(x, y)=U(x)V(y)$), and which equals unity if, and only if, the two variates are in functional relationship. In his definition the author is guided by the form of Pearson's correlation coefficient; however, the definition ensures invariance relative to any transformations of the axes, taken separately. According to the new definition

$$(1)\quad K^2=\text{maximum}\ \frac{\left\{\int\int f(x)g(y)w(x, y)dxdy\right\}^2}{\int\int f^2(x)w(x, y)dxdy\cdot\int\int g^2(y)w(x, y)dxdy},$$

where the maximum is relative to all functions $f(x)$ and $g(y)$ such that

$$(2)\quad \int\int f(x)w(x, y)dxdy=0=\int\int g(y)w(x, y)dxdy.$$

Using Schwartz' inequality it is readily seen that the integral in (1) never exceeds unity; it equals 1 if $f(x)=g(y)=1$, and therefore condition (2) is introduced. This condition moreover ensures that $K=0$ if $w(x, y)=U(x)V(y)$. We obtain the square of Pearson's correlation coefficient r from (1) if we restrict $f(x)$ and $g(y)$ to be linear functions. Hence $r^2\leqq K^2\leqq 1$. Again using Schwartz' inequality the author shows that K^2 can be defined also by

$$(3)\quad K^2=\text{maximum}\int\frac{\left\{\int g(y)w(x, y)dy\right\}^2}{\int w(x, y)dy\int\int g^2(\xi, \eta)w(\xi, \eta)d\xi d\eta}dx,$$

or the corresponding relation using $f(x)$; here $f(x)$ and $g(y)$ are restricted by the condition (2). If, in (3), $g(y)$ is assumed to be a linear function, we obtain as maximum Pearson's correlation ratio K^2_{yx}. Hence $K^2\geqq K^2_{yx}$ and, similarly, $K^2\geqq K_{xy}{}^2$. Using (3) it is easy to find an integral equation of Fredholm type for either $f(x)$ or $g(y)$; the value K^2 is the smallest characteristic value of this equation. The author studies this equation extensively. However, no explicit formulas for K^2 in terms of $w(x, y)$ are given. In a last chapter the author applies his new notion to the theory of errors. *W. Feller* (Providence, R. I.).

Kendall, M. G. On seminvariant statistics. Ann. Eugenics 11, 300–305 (1942). [MF 7076]

Kendall [Ann. Eugenics 10, 215–222 (1940); these Rev. 2, 110], using certain moment-generating operators, verified Fisher's rule for determining sampling semi-invariants of k-statistics. In that paper several rules were set up for applying the operators. In the present paper it is shown that several of the rules hold for any family of algebraic statistics ω_p, where ω_p is homogeneous and of degree p in the observations. *S. S. Wilks* (Princeton, N. J.).

Day, Besse B. and Sandomire, Marion M. Use of the discriminant function for more than two groups. J. Amer. Statist. Assoc. 37, 461–472 (1942). [MF 7536]

This is an application of R. A. Fisher's linear discriminant function to the problem of separating Whitetail deer into age classes on the basis of measurements of weight, certain body dimensions, and four antler characteristics. To choose the best set of these quantities to include in such a discriminant function, a method used by Barnard [Ann. Eugenics 6, 352–371 (1935)] was followed. Comparison of results, particularly with respect to the chance of misclassification, for several choices of single quantities and combinations of them is made. There is some discussion of further theoretical statistical investigations, the need for which is shown by this paper. *C. C. Craig.*

Satterthwaite, Franklin E. Linear restrictions on chi-square. Ann. Math. Statistics 13, 326–331 (1942). [MF 7242]

The statistic χ^2 is usually of the form of the sum of squares of independent components χ_i, $i=1, 2, \cdots, n$, each χ_i being normally distributed with zero mean and unit variance, the χ_i's being subject to $m<n$ algebraically independent linear homogeneous restrictions. The author derives the distribution of χ^2 in the case the restrictions are allowed to be nonhomogeneous by both geometric and analytic methods and shows that, subject to an easily calculated correction, χ^2 still obeys the same distribution function as before with the same loss in degrees of freedom. The fitting of an observed distribution of a continuous variable without correction of the calculated moments for groupings furnishes an interesting illustration. *C. C. Craig.*

Mann, H. B. and Wald, A. On the choice of the number of class intervals in the application of the chi square test. Ann. Math. Statistics 13, 306–317 (1942). [MF 7240]

Pearson's original χ^2 function for testing whether a sample has been drawn from a population with a specified probability distribution is

$$\chi^2=\sum_{i=1}^{k}(a_i-Np_i)^2/Np_i,$$

where k is the number of class intervals into which the range of the variable is divided; p_i is the probability associated with the ith interval and a_i is the number of cases in a sample of size N which fall in the ith interval. It is well known that, if the p_i are all different from zero, then the limiting distribution of χ^2 as $N\to\infty$ is the χ^2 distribution with $k-1$ degrees of freedom. In the present paper the problem of determining the "best" value of k "in the limit" is considered. In order to define "best in the limit" it is necessary to make several definitions: (1) The distance Δ between two distribution functions is the least upper bound of the absolute difference between the two cumulative distributions. (2) Let $f(N, k, \Delta)$ be the greatest lower bound of the power of the χ^2-test with sample size N and number of class intervals k with respect to the class of alternative distributions with distance not less than Δ from the null-hypothesis. (3) A positive integer k is called "best" if there is a Δ so that $f(N, k, \Delta)=\frac{1}{2}$ and $f(N, k', \Delta)\leqq\frac{1}{2}$ for $k'\neq k$. A sequence $\{k_N\}$ of positive integers is called "best in the limit" if $\lim_{N\to\infty}\epsilon(k_N, N)=0$, where $\epsilon(k_N, N)$ is the smallest positive number on the interval $(0, 1)$ for which there exists a Δ such that $f(N, k_N, \Delta)\geqq\frac{1}{2}-\epsilon(k_N, N)$ and $f(N, k', \Delta)\leqq\frac{1}{2}+\epsilon(k_N, N)$ for any positive integer $k'\neq k_N$.

It is shown that, if

$$k_N=4(2(N-1)^2/c^2)^{1/5},$$

where c is determined so that

$$(2\pi)^{-\frac{1}{2}}\int_c^\infty e^{-\frac{1}{2}x^2}dx$$

is equal to probability associated with critical region under the null hypothesis (that is, level of significance), then $\{k_N\}$ is best in the limit. For $N\geqq450$ for 5 percent level of significance or $N\geqq300$ for 1 percent level of significance the authors show that for practical purposes k_N may be considered as a best k. They surmise that the formula for k_N will give a best k, for practical purposes, for $N\geqq200$ on even smaller values of N. *S. S. Wilks.*

Bhattacharyya, D. P. and Narayan, Ram Deva. Moments of the D^2-statistic for populations with unequal dispersions. Sankhyā 5, 401–412 (1941). [MF 7357]

The authors deal with the problem of determining the moments of the D^2 statistic for the case of samples from two normal p-variate populations for each of which the variates are independent. More specifically, let a_i be the mean and s_i^2 be the usual unbiased estimate of the variance of the ith variate in samples of size n from a p-variate normal population π on which the variates are independent. Let a_i' and $s_i'^2$ be similar quantities for a sample of size n' from a p-variate normal p-variate population π' in which the variates are independent. Let

$$D^2=p^{-1}\sum_{i=1}^{p}D_i^2,\quad\text{where}\quad D_i^2=(a_i-a_i')^2/(ks_i^2+k's_i'^2),$$

and where k and k' are any two positive constants. The authors determine recurrence formulas for the moments of D, actually writing down the expressions for the first four moments of D_i^2 from which the first four moments of D can be obtained. It is then shown that the limiting distribution of $[D^2-E(D^2)]/\sigma_{D^2}$, as $n, n'\to\infty$, is normal with zero mean and unit variance. *S. S. Wilks.*

Roy, S. N. and Bose, R. C. The use and distribution of the Studentized D^2-statistic when the variances and covariances are based on k samples. Sankhyā 4, 535–542 (1940). [MF 7358]

The sampling theory of the D^2-statistic for the case of two samples from normal p-variate populations with identical variance-covariance matrices has been treated in detail by Mahalanobis and Bose. In this case

$$D^2=p^{-1}\sum_{i,j=1}^{p}c^{ij}(a_i-a_i')(a_j-a_j'),$$

where $\|c^{ij}\|$ is the inverse of the matrix of the weighted averages of sample variances and covariances (that is, $c_{ij}=(na_{ij}+n'a'_{ij})/(n+n')$, where n and n' are the sample sizes and a_{ij} and a'_{ij} are the variances and covariances in the two samples). In the present paper, the authors consider k samples from normal p-variate populations with identical variance-covariance matrices. They determine the distribution of any given D_{rs}^2 $(r\neq s)$, where

$$D_{rs}^2=p^{-1}\sum_{i,j=1}^{p}c^{ij}({}_ra_i-{}_sa_i)({}_ra_j-{}_sa_j),$$

where $\|c^{ij}\|$ is the inverse of the matrix $\|c_{ij}\|$, where

$$c_{ij}=\Big(\sum_{r=1}^{k}n_r\,{}_ra_{ij}\Big)\Big/\Big(\sum_{r=1}^{k}n_r\Big),$$

$\|{}_ra_{ij}\|$ being the variance-covariance matrix in the rth sample, ${}_ra_i$ being the mean of the ith variable in the rth sample. *S. S. Wilks* (Princeton, N. J.).

Smith, B. Babington. Note on an alternant suggested by statistical theory. Edinburgh Math. Notes no. 32, 19–22 (1941). [MF 7550]

The alternant here considered was suggested by a square array which arose in work on the distribution of Spearman's coefficient of rank correlation [see Biometrika 30, 251–273 (1938–39)]. Two means of evaluating the alternant are shown, one suggested by Turnbull and the other by Rutherford. *C. C. Craig* (Ann Arbor, Mich.).

Wilson, Edwin B. and Worcester, Jane. Contingency tables. Proc. Nat. Acad. Sci. U. S. A. 28, 378–384 (1942). [MF 7187]

The paper aims to derive general principles for contingency tables so that significance tests, involving chi-square, may be applied when it is difficult to set up random sampling procedure or to carry out the computations. The notation of Yule is used in which, if A denotes a character, α denotes its absence. Also, $n_1=(AB)$ is the number of individuals having both characters A and B; $n_2=(\alpha B)$, $n_3=(A\beta)$, $n_4=(\alpha\beta)$. A sample of N individuals is supposed drawn from a universe in which $p_1=p_{AB}$ is the probability that the individual drawn will have both characters; $p_2=p_{\alpha B}$, $p_3=p_{A\beta}$, $p_4=p_{\alpha\beta}$. These p's are constant; so also $p_A=p_1+p_3$, $p_B=p_1+p_2$. If, for a four-fold table, the marginal totals are given, only one of the four frequencies,

say n_1, may be assigned. The set of all such tables, obtained by varying n_1, is considered. The condition $p_1p_4=p_2p_3$ is found to imply $p_1=p_Ap_B$. In this case, the probability of an individual table is proportional to $[n_1!n_2!n_3!n_4!]^{-1}$. Also, it is thus proportional if drawing is made from two different point-binomial universes with probabilities $p_1=p$, $p_2=1-p$, $p_3=p'$, $p_4=1-p'$, in the two pairs of cells coming from the two universes independently. This principle is extended to the case of ν cellular universes with unknown but fixed probabilities. The authors state: ". . . the principle gives a solution for the relative probabilities and for the expected table and, if one is willing to accept that solution and the further rule that the significance of a table in the spread of two degrees of freedom is to be determined by the sum of the probabilities of the table and of all tables no more probable, the two rules taken together will give a test of significance." *E. L. Dodd* (Austin, Tex.).

Wilson, Edwin B. and Worcester, Jane. The association of three attributes. Proc. Nat. Acad. Sci. U. S. A. **28**, 384–390 (1942). [MF 7188]

In continuation of the paper reviewed above, three interrelated characters A, B and C are considered. It takes six numbers (A), (B), etc., to describe the frequencies of A, B, C, α, β and γ, individually; twelve numbers for pairs; and eight numbers for sets of three. "The chance of drawing the particular sample is

$$(1)\qquad P=\frac{N!p_{ABC}^{(ABC)}\cdots p_{\alpha\beta\gamma}^{(\alpha\beta\gamma)}}{(ABC)!\cdots(\alpha\beta\gamma)!}.$$

When discussing the statistical significance of the sample, which may be considered as a $2\times2\times2$ table, certain values taken from the sample must be "used." There are four cases which we propose to discuss: (1) N, (A), (B), (C), the grand total and the "edge" subtotals; (2) (AB), (αB), $(A\beta)$, $(\alpha\beta)$, (C), (γ), one marginal "face" and the complimentary edge; (3) two marginal faces, those of A, B and A, C; (4) all three marginal faces. In the discussion it will be assumed that the sample probability (1) must be independent of the values of the unknown cell probabilities for all variations of the sample consistent with the marginal totals used." *E. L. Dodd* (Austin, Tex.).

Geiringer, Hilda. Observations of analysis of variance theory. Ann. Math. Statistics **13**, 350–369 (1942). [MF 7245]

To distinguish between the series of Lexis, Bernoulli and Poisson one compares the values of three test functions which are quadratic forms in the observed values of the variables. The Bernoulli series is characterized by the equality of these functions, whereas the series of Poisson and Lexis are characterized by inequalities. The author considers three additional quadratic forms and shows that among the six forms there are characteristic equalities as well as inequalities for the series of Lexis and Poisson. These results are based on the assumption of independence. In the latter part of the paper the author studies the effect on the above relations of dependence of the variables.

A. H. Copeland (Ann Arbor, Mich.).

Irwin, J. O. On the distribution of a weighted estimate of variance and on analysis of variance in certain cases of unequal weighting. J. Roy. Statist. Soc. (N.S.) **105**, 115–118 (1942). [MF 7368]

"Let x_i $(i=1, 2, \cdots, n)$ be a set of variates which are normally and independently distributed with variances $1/w_i$ and let

$$\bar{x}=\sum_{j=1}^{n}w_jx_j/\sum_{j=1}^{n}w_j$$

be their weighted mean. It is well known and of frequent application that the quantity

$$(*)\qquad \sum_{j=1}^{n}w_j(x_j-\bar{x})^2$$

is distributed as χ^2 with $(n-1)$ degrees of freedom, but there does not seem to be an explicit proof in the literature." Such a proof is now furnished. The author writes $(*)$ in a form "in which each term is the square of the difference between a weighted mean and a single observation, divided by its variance. This result may be utilized to apply analysis of variance methods to certain cases of unequal numbers or unequal weights." *E. L. Dodd* (Austin, Tex.).

Roy, S. N. Analysis of variance for multivariate normal populations: the sampling distribution of the requisite p-statistics on the null and non-null hypotheses. Sankhyā **6**, 35–50 (1942). [MF 7364]

Let $\Sigma_1, \cdots, \Sigma_r$ be r p-variate normal populations and let S_β $(\beta=1, \cdots, r)$ be a sample of size N_β drawn from the population Σ_β. Denote by $x_{i\beta\alpha}$ $(i=1, \cdots, p;\ \beta=1, \cdots, r;\ \alpha=1, \cdots, N_\beta)$ the αth observation on the ith variate in the sample S_β. The populations $\Sigma_1, \cdots, \Sigma_r$ are assumed to have a common set of variances and covariances but not necessarily a common set of mean values. Let $\sum_{\beta=1}^{r}N_\beta=N$; $x_{i\beta}=(\sum_\alpha x_{i\beta\alpha})/N_\beta$; $x_i=(\sum_\beta\sum_\alpha x_{i\beta\alpha})/N$;

$$a'_{ij}=[\textstyle\sum_\beta N_\beta(x_{i\beta}-x_i)(x_{j\beta}-x_j)]/(r-1)$$

and

$$a''_{ij}=[\textstyle\sum_\beta\sum_\alpha(x_{i\beta\alpha}-x_{i\beta})(x_{j\beta\alpha}-x_{j\beta})]/(N-r).$$

The author considers the determinantal equation

$$|a'_{ij}-ta''_{ij}|=0$$

and derives the joint distribution of the roots in both cases when the populations $\Sigma_1, \cdots, \Sigma_r$ have a common set of mean values and when they do not have a common set of mean values. This distribution is of importance in the analysis of variance for multivariate normal populations.

A. Wald (New York, N. Y.).

Roy, S. N. The sampling distribution of p-statistics and certain allied statistics of the non-null hypothesis. Sankhyā **6**, 15–34 (1942). [MF 7363]

Let Σ' and Σ'' be two p-variate normal populations with covariance matrices $\|\alpha'_{ij}\|$ and $\|\alpha''_{ij}\|$ $(i, j=1, \cdots, p)$. A sample of size N_1 is drawn from population Σ' and a sample of size N_2 from Σ''. Denote by $x'_{i\alpha}$ $(i=1, \cdots, p;\ \alpha=1, \cdots, N_1)$ the αth observation on the ith variate in the first sample, and by $x''_{i\alpha}$ $(i=1, \cdots, p;\ \alpha=1, \cdots, N_2)$ the αth observation on the ith variate in the second sample. Let $a'_i=(x'_{i1}+\cdots+x'_{iN_1})/N_1$, $a''_i=(x''_{i1}+\cdots+x''_{iN_2})/N_2$,

$$a'_{ij}=(\sum_{\alpha=1}^{N_1}(x'_{i\alpha}-a'_i)(x'_{j\alpha}-a'_j))/(N_1-1)$$

and

$$a''_{ij}=(\sum_{\alpha=1}^{N_2}(x''_{i\alpha}-a''_i)(x''_{j\alpha}-a''_j))/(N_2-1).$$

The author considers the determinantal equation

$$|a'_{ij}-ka''_{ij}|=0$$

in k and derives the joint distribution of roots. This is an

extension of some previous results of R. A. Fisher [Ann. Eugenics 9, 238–249 (1939); these Rev. 1, 248] and P. L. Hsu [Ann. Eugenics 9, 250–258 (1939); these Rev. 1, 248], who obtained the distribution in question in the special case when $\|a'_{ij}\| = \|a''_{ij}\|$. The roots of the determinantal equation under consideration are of importance for purposes of discrimination between multivariate normal populations with respect to covariances. *A. Wald.*

Kendall, M. G. Note on the estimation of a ranking. J. Roy. Statist. Soc. (N.S.) 105, 119–121 (1942). [MF 7369]

Suppose n objects are ranked m times, such that for each ranking there is a one-to-one correspondence between the objects and the positive integers 1 to n. Friedman, Kendall and Smith have done work on testing for significant concordance among the m rankings. In the present note Kendall considers the problem of estimating "true ranks" when significant concordance exists. If S_j denotes the sum of the ranks for the jth object, Kendall proposes the ranking of the S_j as the estimates of "true ranks," verifying that they minimize the sum of squares $\sum_1^n (S_j - mX_j)^2$ and maximize the average Spearman rank correlation coefficient for the m rankings, where X_j is the true rank of the jth object. *S. S. Wilks* (Princeton, N. J.).

Kosambi, D. D. A test of significance for multiple observations. Current Sci. 11, 271–274 (1942). [MF 7154]

For the purpose of discriminating multivariate normal populations with respect to their mean values, test functions have been introduced and studied by H. Hotelling, R. A. Fisher, P. C. Mahalanobis, R. C. Bose, S. N. Roy and others. If the number of variates as well as the number of populations is greater than two, the application of the test would require the knowledge of certain probability distributions which have not yet been tabulated. The author proposes the following test procedure: Let $x_1, \cdots, x_p$ be the variates in the multivariate populations under consideration and let $x_{i\beta\alpha}$ be the αth observation on x_i drawn from the βth population. We replace the p variates $x_1, \cdots, x_p$ by a single variate $y = \lambda_1 x_1 + \cdots + \lambda_p x_p$ and let $y_{\beta\alpha} = \sum_{i=1}^p \lambda_i x_{i\beta\alpha}$. Considering merely the variate y, we calculate the well-known F-statistic (analysis of variance statistic) for discriminating univariate populations. For discriminating the multivariate populations it is proposed that the statistic F^*, which is the maximum of F with respect to $\lambda_1, \cdots, \lambda_p$, be used. The author states that F^* has the ordinary F-distribution tabulated by R. A. Fisher and others. It seems to the reviewer that this statement of the author would be correct only if the coefficients $\lambda_1, \cdots, \lambda_p$ were chosen independently of the sample. Since $\lambda_1, \cdots, \lambda_p$ are functions of the sample values, the sampling distribution of F^* will arise partly from the sampling variation of $\lambda_1, \cdots, \lambda_p$ and consequently the distribution of F^* need not be the same as that of F. *A. Wald.*

Jeffreys, Harold. On the significance tests for the introduction of new functions to represent measures. Proc. Roy. Soc. London. Ser. A. 180, 256–268 (1942). [MF 6930]

In his book [Theory of Probability, Oxford University Press, Oxford, 1939; cf. these Rev. 1, 151] H. Jeffreys has developed a theory of tests of significance based on a subjective notion of probability. In this paper the exact formulae for some tests of significance are reduced to single integrals and their evaluation is carried out. The formulae in question are based on the assumption that, if nothing is known about the parameter which may take any value in a finite range, or from $-\infty$ to $+\infty$, its prior probability is uniformly distributed. If the parameter is restricted to the range $(0, \infty)$, the prior probability of its logarithm is assumed to be uniformly distributed. Most of the modern statisticians will consider these assumptions as rather arbitrary and hardly acceptable. *A. Wald* (New York, N. Y.).

Scheffé, Henry. On the theory of testing composite hypotheses with one constraint. Ann. Math. Statistics 13, 280–293 (1942). [MF 7238]

This paper is a contribution to the Neyman-Pearson theory of testing hypotheses in the case that the distribution law governing the universe sampled is a function of several parameters. The hypothesis to be tested concerns the value of a single one of these parameters; the others are "nuisance" parameters. The regions of type B defined by Neyman for one nuisance parameter are extended to the present case and the first theorem, giving one of the two principal results, embodies a method for determining such regions. The conditions and methods are modeled on those of Neyman and Pearson but the analysis is more extended and deals with points of some importance not cleared up by Neyman. One condition involves a moment problem which is considered in the appendix in the case of a multivariate normal universe. The second principal result gives sufficient conditions that a region of type B be also of type B_1, a region of type B_1 being the natural extension of the region of type A_1 of Neyman and Pearson to allow for the presence of nuisance parameters. *C. C. Craig* (Ann Arbor, Mich.).

Wolfowitz, J. Additive partition functions and a class of statistical hypotheses. Ann. Math. Statistics 13, 247–279 (1942). [MF 7237]

This paper contains important contributions to the theory of testing statistical hypotheses when the distribution functions of the random variables under consideration are entirely unknown except for continuity. One of the problems treated is the following: Let n_1 and n_2 random independent observations be made, respectively, on two populations whose distribution functions are entirely unknown except for continuity. It is desired to test whether the two distributions are identical. The author extends the Neyman-Pearson likelihood ratio criterion to this problem as follows: Ω is the totality of all couples of continuous distribution functions; ω is that subset of Ω for which the two members of a couple are identical. The sample space consists not of the observations themselves but of sequences V which characterize the ranking in order of size of the observations from the two populations. Under the hypothesis to be tested all these sequences have the same probability. Each element of Ω assigns to each V a probability; let $T(V)$ be the maximum of these in Ω. The order in which the sequences V are to be chosen for the critical region is $V_1, V_2, \cdots, V_f$, where $T(V_1) \geqq T(V_2) \geqq \cdots \geqq T(V_f)$. A sufficient number of V's is chosen to make the critical region of proper size. This method is also applied to the problem of testing whether two variates are independently distributed, nothing being known about their joint distribution function except its continuity. The n observation couples are ranked in the order of size of observations on one of the variates. The sample is then characterized by a permutation of the first n integers which are the ranks of the observations on the

other variate. The sample space is the totality of these permutations.

To facilitate application of the theory the author presents several new results on additive partition functions, of which the following is a typical example. Let $f(x)$ be a function defined for all positive integral x and subject only to the following restrictions: (1) There exists a pair of positive integers, a and b, such that $f(a)/f(b) \neq a/b$. (2) The series $\sum_{i=1}^{\infty} |f(i)| 2^{-i}$ converges. Let $A=(a_1, a_2, \cdots, a_s)$ be any sequence of positive integers whose sum is a given integer n. Two sequences A which contain the same elements in a different order are to be considered different. Let $F(A)=\sum_{i=1}^{s} F(a_i)$. Then there exist, for every positive integer n, two numbers E_n and σ_n, such that 2^{-n+1}, multiplied by the number of sequences A for which the inequality $(F(A)-E_n)<y\sigma_n$ holds, approaches $(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2}dz$, as $n\to\infty$, where y is any real number. *A. Wald.*

Dodd, Edward L. Certain tests for randomness applied to data grouped into small sets. Econometrica **10**, 249–257 (1942). [MF 7183]

The author has devised four simple tests of randomness of digits applicable, for example, in the case of a decimal system to Tippett's table of random numbers: (1) An order test based on pairs of integers in which a given pair of integers ab is classed as "up" (for a decimal system) if $a<b$ or $a=b=5, 6, 7, 8, 9$ and "down" otherwise, under the hypothesis of randomness $P(\text{up})=P(\text{down})=\frac{1}{2}$. (2) An order test based on triples of integers such that, for a decimal system of integers, a, b, c is designated as "up" if $a\leqq b \leqq c (a<c)$, or $a=b=c\geqq 5$, "down" if $a\geqq b\geqq c (a>c)$ or $a=b=c<5$, "max" if $a<b>c$, and "min" if $a>b<c$; under hypothesis of randomness $P(\text{up})=P(\text{down})=.215$, $P(\text{max})=P(\text{min})=.285$. (3) A range test based on the range of a set of k integers, in which the range r is defined as the difference between the least and greatest integers in the set. Under the hypothesis of randomness in a decimal system, Dodd gives probabilities of the various values of $r(0, 1, 2, \cdots, 9)$ for $k=2, 3, 4, 5, 6$. (4) A replication test in which the probability is given that a set of k integers will contain s doubles, t triples, u quadruples, v quintuples, w sextuples. Numerical values of probabilities are given for the hypothesis of randomness of numbers in a decimal system and for $k=5$. *S. S. Wilks* (Princeton, N. J.).

Savur, S. R. A test of significance in approximate periodogram analysis. Sankhyā **6**, 77–84 (1942). [MF 7365]

To test a sequence of $N=nk$ items for a period of n, the author supposes that the N items are set down in the usual table with n columns and k rows; the arithmetic mean of each column is supposed found. The difference between the least and the greatest of these means "is taken to be approximately equal to twice the amplitude of the period n, if any, in our data." Whether this difference is significant remains to be tested. For the distribution of this difference the author finds an analytic expression, also for the probability that a specified difference will not fall below a specified number. The integration prescribed appears difficult, so the author approximates integrals by sums which can be found with the aid of Table II of Karl Pearson's Tables for Statisticians and Biometricians, Part I. Reference is made to Walker's criterion and to R. A. Fisher's correction and a comparison is made with a test of Egon S. Pearson. *E. L. Dodd* (Austin, Tex.).

Nair, K. R. The median in tests by randomization. Sankhyā **4**, 543–550 (1940). [MF 7359]

The use of tests based on the randomization principle in order to avoid assumptions concerning the nature of the population sampled has been discussed by various writers [see E. S. Pearson, Biometrika **30**, 53–64 (1938) or E. J. G. Pitman, Suppl. J. Roy. Statist. Soc. **4**, 119–130 (1937)]. Such tests in the case of two paired or two independent samples of equal size have heretofore been based on the mean of pair differences in the first case or the difference of means for each pair of samples in the second case. The distribution of these means or differences of means in the subpopulation formed by randomization is very difficult to find. The author shows that the distribution of the median in the case of an odd number of observations, or of one of two central values if there is an even number of observations in each sample, comes out simply. The distribution of range or mid-range is also easily found. A numerical illustration is provided for an example used by E. S. Pearson [loc. cit.]. *C. C. Craig* (Ann Arbor, Mich.).

Nair, K. R. A note on the method of "fitting of constants" for analysis of non-orthogonal data arranged in a double classification. Sankhyā **5**, 317–328 (1941). [MF 7353]

In the analysis of variance for agricultural experiments, a simple case arises when in every block each treatment occurs once and only once, and vice versa. On account of some mishap, however, certain plots may have to be excluded, and among the different designs for experiment are some in which incompleteness is deliberately imposed. The author discusses only the situations in which the data fall in two classes, say block and treatments. The Latin square designs are not included. The assumption is made that the effects of the two classes are independent and additive, and that there is also an independent additive normally distributed variation from the hypothetical mean of all measurements. The sum of the squares of these variations is then minimized, subject to two linear restraints, to get efficient estimates.

The author's summary is as follows: "A convenient procedure has been evolved to get the sum of squares of the various components of the analysis of variance table when nonorthogonal data in double classification are analyzed by the fitting of constants. The unification it brings to the analyses of field experiments in randomized block for testing a single factor, and having various types of non-orthogonality, has been indicated. The problem of tests of significance of difference between any two treatment effects has been discussed with special reference to Yates' balanced incomplete block designs having (i) a missing plot and (ii) a missing block." *E. L. Dodd* (Austin, Tex.).

Nair, K. R. The application of the technique of analysis of covariance to field experiments with several missing or mixed-up plots. Sankhyā **4**, 581–588 (1940). [MF 7362]

Kishen, K. Symmetrical unequal block arrangements. Sankhyā **5**, 329–344 (1941). [MF 7354]

Kishen, K. On a simplified method of expressing the components of the second order interaction in a 3^3 factorial design. Sankhyā **4**, 577–580 (1940). [MF 7361]

Sukhatme, P. V. On Bernstein's improved method of estimating blood-group gene frequencies. Sankhyā **6**, 85–92 (1942). [MF 7366]

Kosambi, D. D. On valid tests of linguistic hypotheses. New Indian Antiquary **5**, 21–24 (1942). [MF 7152]

The Harvard philologist F. K. Zipf has proposed the law that the number of words used n times in a connected piece of writing is proportional to n^{-2}. The author raises serious objections to this law on the basis of results he obtained by applying modern statistical methods to some data.

A. Wald (New York, N. Y.).

Kimball, Bradford F. General theory of plant account subject to constant mortality law of retirements. Econometrica **11**, 61–82 (1943). [MF 7787]

A thorough investigation of the problem of plant depreciation based on the mathematical theory of self-renewing aggregates and using in particular results of a paper by the reviewer [Ann. Math. Statistics **12**, 243–267 (1941); these Rev. **3**, 151]. *W. Feller* (Providence, R. I.).

Lange, Oscar. Theoretical derivation of elasticities of demand and supply: the direct method. Econometrica **10**, 193–214 (1942). [MF 7181]

The elasticity Eu/Ev of a variable u with respect to another variable v is defined by $Eu/Ev=(\partial u/\partial v)(v/u)$. The elasticities of certain economic variables with respect to prices, such as the elasticity of demand for a factor of production, the elasticity of supply of a product, the elasticity of demand for a consumption good, etc., appear in the literature in a rather complicated and awkward form. In this paper the author shows that the derivation of the elasticity formulas can be considerably shortened and the final results can be put in a much simpler and more elegant form by using a direct method of derivation based on the algorithm of the calculus of elasticities. The direct method leads also to a simple formula for the so-called partial elasticity of substitution. In the last part of the paper the author generalizes the calculus of elasticities to the case of matrices and vectors and makes further simplifications in the derivation of the elasticity formulas. *A. Wald.*

Wolfenden, Hugh H. On the formulae for calculating the "exposed to risk" in constructing mortality and other tables from the individual records of insured lives. Trans. Actuar. Soc. America **43**, 234–277 (1942). [MF 7563]

GEOMETRY

***Busemann, Herbert. Metric Methods in Finsler Spaces and in the Foundations of Geometry.** Annals of Mathematics Studies, no. 8. Princeton University Press, Princeton, N. J., 1942. viii+243 pp. $3.00.

This book provides an excellent introduction to the developing subject of metric geometry of manifolds. The general theory of metric properties of spaces which was developed by Menger and is now classical, shows its fruitfulness in giving a basis for more detailed investigations of locally Euclidean spaces. The book treats spaces of finite dimensions and presents most results for the two dimensional case. A very interesting confluence of topological, group theoretical, and differential geometry methods is exhibited by the author, whose own results form a considerable part of the material.

The first chapter deals with general metric spaces in which geodesics exist. One obtains the description of the topological structure of two such dimensional spaces. In the second chapter, further conditions are imposed on the space, making it a general Finsler space, the natural generalization of a Riemannian space. Finsler spaces may be defined as locally Minkowskian, that is to say, such that sufficiently small neighborhoods are isometric to a neighborhood of Euclidean space metrized by a symmetric convex gauge. One of the most important aspects of the book is that it permits a general development of a theory without the differentiability assumptions, and so supplements and generalizes the more classical investigations of Riemann's spaces. A theorem is proved to the effect that a general space with geodesics is necessarily a Finsler space if certain sequences of geodesics converge. In the third chapter the spaces are specialized further by the requirement that there should exist at least one (then unique) geodesic through two given points. It is shown that geodesics are then congruent either to straight lines or to circles. It is proved that the general two dimensional space with unique geodesics is homeomorphic to either the Euclidean plane or to the projective plane. The important problem of characterizing all families of curves in the plane which can be regarded as straight lines under a suitable metric is solved. This is an interesting case of the solution of the inverse problem of the calculus of variations. Here again the problem of generalizing the result to the projective plane and to higher dimensions is still open. A further discussion of the properties of parallels gives an insight into the problems of foundations of geometry from the point of view of metric geometry. In the fourth chapter spaces with convex spheres are studied. This additional property is proved equivalent to the existence of perpendiculars. Minkowskian n-dimensional, $n\geqq 3$, geometries are characterized through the use of this notion. Here the problem of obtaining this theorem of characterization for second dimensional spaces seems to be difficult! In the fifth, perhaps least complete chapter, motions or isometric mappings are studied. It is proved that a space with unique geodesics is homogeneous (that is, if two of its subsets are congruents, then they can be made to coincide through a motion of the whole space), if triplets of points of small diameter can be moved into each other. (The conjecture that homogeneity follows from the movability of couples of points alone is disproved by an example of the author developed from an idea of Fubini.) All plane geometries with a transitive group of motions are determined. The general discussion of the groups of motions remains rather incomplete.

The book is self contained and very readable. Its great value, in the reviewer's opinion, lies in the fact that it gives a systematic investigation of the role of differentiability in foundations of the theory of Riemannian spaces, in its characterizations of Minkowskian and hyperbolic geometries, and in the introduction of a class of natural problems (mostly unsolved for higher dimensions), which may guide the further development of the theory. It may perhaps stimulate the undertaking of similar work of characterization of metrics in infinitely dimensional vector spaces.

S. M. Ulam (Madison, Wis.).

Baer, Reinhold. A unified theory of projective spaces and finite abelian groups. Trans. Amer. Math. Soc. **52**, 283–343 (1942). [MF 7122]

Here the author presents a theory which explains the relation between projective geometry and finite Abelian groups [cf. Carmichael, Groups of Finite Order. Ginn and

Co., Boston, 1937]. The paper is divided into three parts: (I) Synthetic Theory; (II) Analytic Theory; (III) Construction of the Underlying Group.

Part (I) begins with an account of the required properties of Dedekind sets and introduces the notion of a "cycle" and its order. Cycles of order 1 correspond, on the one hand, to points of projective space and, on the other, to cyclic subgroups of prime power order of a finite Abelian group. For such correspondences it is necessary and sufficient to require that (1) every element of the Dedekind set is a sum of cycles, and (2) a quotient system is a cycle if and only if it contains at most two different smallest elements not zero.

In part (II) the concept of an Abelian operator group is introduced; the following definitions give the key-note of the discussion. (i) If E is a set of endomorphisms of the Abelian group G, S a subset of G, then S is said to be E-admissible if $SE \leqq S$. The system $D(G; E)$ of all E-admissible subgroups of G is a ring and consequently also a Dedekind set. On the other hand, (ii) if L be a set of subgroups of G the endomorphism e of G is L-admissible if $Se \leqq S$ for every subgroup S in L. The ring L of subgroups of G is primary if every cyclic subgroup gL in L is a cycle in L. In theorem II.1.2 the author proves that, if such a primary L contains either no subcycle of order n or at least three independent ones, then L is the set $D(G; E)$ of all E-admissible subgroups of G, where E is the ring of all L-admissible endomorphisms of G. By considering an isomorphism of the Abelian group G upon another Abelian group H the author is led to the fundamental theorem of projective geometry and thence to Pappus' theorem. The proof of Pappus' theorem [cf. Baker, Principles of Geometry, vol. I, Cambridge (England) 1922] is particularly interesting in this setting since it flows so naturally from the ideas already developed.

Part (III) begins with a discussion of Desargues' theorem and its relation to vector addition. The constructions involved are necessarily complicated but, as the author suggests, this section is of interest apart from its application to the problem in hand. This problem is the determination of necessary and sufficient conditions that the parts of an element g in a partially ordered set satisfying $(D, 6)$ should be identifiable with the admissible subgroups of some "primary Abelian operator group." The condition $(D, 6)$ is: If g contains one subcycle of order n, then g contains at least 6 independent subcycles of order n. The number of independent subcycles must be at least 4, since Desargues' theorem cannot be proved in the plane; whether it is necessary to go as high as 6 remains an open question.

G. de B. Robinson (Ottawa, Ont.).

Turnbull, H. W. The projective invariants of four medials. Proc. Edinburgh Math. Soc. (2) **7**, 55–72 (1942). [MF 7495]

A medial is a linear subspace of dimension $k-1$ of a projective space of dimension $2k-1$. The author investigates these by using an "outer multiplication" of the $k \times 2k$ matrices defining the medials. Projective invariants X_i of a set of four medials are constructed and are shown to be the coefficients of the characteristic equation of a collineation associated with the medials. The roots of this equation are cross-ratios of the sets cut out by the medials (assumed in general position) on their k secant lines. Other properties of the X_i are discussed. *R. J. Walker.*

Ward, James A. Note on the quartic and its Hessian. Nat. Math. Mag. **17**, 165–167 (1943). [MF 7777]

Scott, T. A dual quadratic transformation associated with the Hessian conics of a pencil. Edinburgh Math. Notes no. **32**, 14–16 (1941). [MF 7548]

It is shown that in a pencil of conics there are two curves f_1 and f_2 such that for any third conic f_3 of the pencil the invariant Ω_{123} is zero. [The author states that this is true for any pencil, but his proof is valid only for pencils with distinct base points.] A related quadratic line-to-point transformation is constructed. *R. J. Walker.*

Turnbull, H. W. Collapsible circular sections of quadric surfaces. Edinburgh Math. Notes no. **32**, 16–19 (1941). [MF 7549]

Edge, W. L. A type of periodicity of certain quartic surfaces. Proc. Edinburgh Math. Soc. (2) **7**, 73–80 (1942). [MF 7496]

This paper gives further examples of periodicity in a series of contravariant quartic envelopes and of covariant quartic loci first discussed by Reye [J. Reine Angew. Math. **82**, 173–206 (1877), in particular, p. 198]. Consider a form (point locus) of order $2p$ in n homogeneous variables and its pth polar. The matrix μ of its coefficients, associated with proper multinomial numerical factors, is bordered by a row and column of products of the dual coordinates of order p in such sequence as to make the matrix truly symmetric. For $p=1$, this is the well-known polar duality as to a quadric primal. When the process is repeated, the original quadric primal is reproduced. When $p=2$ such recurrence of the original form is not possible except in particular cases. Reye obtained one when the given quartic is the square of a quadric surface ($n=4$). For $n=3$ a similar question was considered by Coble [Trans. Amer. Math. Soc. **4**, 65–85 (1903)] concerning plane quartic curves.

The present paper gives two proper quartic surfaces having the same property. These are

$$x^4+y^4+z^4+t^4+12xyzt=0$$

and the developable formed by the tangents to a space cubic curve. Various properties of polar cubic surfaces of the points of a plane are also given. *V. Snyder.*

Ionescu, D. V. Sur une configuration de six points attachée à un tétraèdre coupé par trois plans parallèles à une face. Mathematica, Timişoara **18**, 104–111 (1942). [MF 7426]

Let ABC, $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be parallel sections of the same trihedral angle S, and let P, P_1, P_2, P_3 be four points collinear with the vertex S and situated in the four planes considered. Let Q_1, Q_1' be the two points determined by the two triads of planes

$$(1)\qquad \begin{matrix}(P_1\|SBC,\ P_2\|SCA,\ P_3\|SAB),\\ (P_1\|SBC,\ P_2\|SAB,\ P_3\|SCA),\end{matrix}$$

where $P_1\|SBC$ represents the plane through the point P_1 parallel to the plane SBC, etc. Circular permutations of the planes SBC, SCA, SAB in (1) determine two analogous pairs of points Q_2, Q_2' and Q_3, Q_3'. The paper studies the six points thus obtained, and the author shows that the six points lie on an ellipse, that the center of this ellipse is the common centroid of the two triangles $Q_1Q_2Q_3$ and $Q_1'Q_2'Q_3'$, etc. The proofs utilize vector analysis throughout.

N. A. Court (Norman, Okla.).

Mandan, Ram. Properties of mutually selfpolar tetrahedra. Bull. Calcutta Math. Soc. **33**, 147–155 (1941). [MF 7160]

Given a tetrahedron $(T)=ABCD$, let a line meet the edges DA, BC in the points P, Q, and let R, S be the harmonic conjugates of P, Q with respect to the pairs of vertices D, A and B, C. Let A', B' be any two points harmonic to P, Q, and C', D' any two points harmonic to R, S. The author points out that each face of the tetrahedron $(T')=A'B'C'D'$ is the tetrahedral polar plane of the respectively opposite vertex for the tetrahedron (T), in the usual sense, and that this relation between (T) and (T') is mutual. However, two such mutually self-polar tetrahedrons are not desmic unless a second pair of opposite edges of (T'), besides the edges $A'B'$, $C'D'$, meet a second pair of opposite edges of (T). The eight vertices of two mutually self-polar tetrahedrons form an associated group if, and only if, the tetrahedrons are desmic. Groups of mutually self-polar tetrahedrons and desmic tetrahedrons are considered, and the relation of such a group of tetrahedrons to Kummer's surface is pointed out. The treatment is synthetic.

N. A. Court (Norman, Okla.).

Hameed, Asghar. On mutually self-polar tetrahedra. Bull. Calcutta Math. Soc. **33**, 157–186 (1941). [MF 7161]

This paper covers practically the same ground as the paper reviewed above. The author also notices that the point A' is a vertex of an infinite number of tetrahedrons (T') mutually self-polar with (T) obtained by making the points C', D' vary on the line RS. Two more analogous groups of such tetrahedrons having A' for a vertex are obtained by considering the other two pairs of opposite edges of (T), besides the pair DA, BC. The transforms of mutually self-polar tetrahedrons by homology, skew homology and polar reciprocation with respect to a quadric are considered. Some of these results for desmic tetrahedrons have been anticipated by the reviewer [Mathesis **51**, 307–313 (1937)].

N. A. Court (Norman, Okla.).

Bilger, Gérard. Remarques sur les polygones et leurs étoilés. Enseignement Math. **38**, 325–329 (1942). [MF 7307]

Let C be a regular polygon, c_i its sides, l its perimeter and S its area; C' is any of the star polygons of C which have the same vertices as C, and c'_i, l', S' refer to the sides, perimeter and area of C'; C'' is the configuration consisting of the lines from the center of C to the vertices. The author states without proof a number of theorems concerning C and C'. Typical of these results is the following. Let M be an exterior point of C, and γ a closed path beginning and ending at M. Then $\sum\pm d_i/l=\sum\pm d'_i/l'=\sum\pm d''_i/l''$ and $\sum\pm d_i^2/S=\sum\pm d_i'^2/S'$, where d_i, etc., are the perpendicular distances from M to the various lines c_i, etc., and the algebraic sign is determined by a rule supplied by the author. The summation extends over the sides cut by the path γ. Some of these results are restated in terms of polyhedra.

J. W. Green (Rochester, N. Y.).

Crain, Karleton W. A locus related to the Euler line. Nat. Math. Mag. **17**, 163–164 (1943). [MF 7776]

Peineke, H. Fehlerbetrachtung zur Rinaldinischen Regel. Z. Angew. Math. Mech. **22**, 116–117 (1942). [MF 7661]

This paper is concerned with Rinaldini's method of dividing a circle into n approximately equal parts.

W. Feller (Providence, R. I.).

Srinivasiengar, C. N. The resultant of wrenches on two given screws. Bull. Calcutta Math. Soc. **34**, 131–138 (1942). [MF 7578]

Considering the methods of determining the resultant of wrenches on two given screws, the author contrasts the "rather indirect" method given by R. S. Ball [Theory of Screws, Cambridge, 1900] with a "direct" method which makes use of the linear complex associated with a system of forces.

W. Prager (Providence, R. I.).

***Flexner, William W. and Walker, Gordon L. Military and Naval Maps and Grids.** Dryden Press, New York, 1942. 96 pp. $1.00.

This is an elementary introduction to the study of the Mercator map, Lambert and stereographic projection, etc. Applications are illustrated by numerous examples.

O. Neugebauer (Providence, R. I.).

Convex Domains, Integral Geometry

Botts, Truman. Convex sets. Amer. Math. Monthly **49**, 527–535 (1942). [MF 7345]

This paper contains two new proofs, one of which the author credits to McShane, of the following theorem: If x_0 is a boundary point of a closed convex set K in Euclidean n-space E_n, then there is a plane of support of K passing through x_0. The new proofs avoid the limiting process which enters in Dines' proof [Amer. Math. Monthly **45**, 199–209 (1938)]. The paper closes with some related theorems, previously treated in the two dimensional case only, on the existence and continuous turning of planes tangent to convex surfaces; the limiting process, avoided in the first theorem, enters in their proofs.

G. B. Price.

Kakutani, Shizuo. A proof that there exists a circumscribing cube around any bounded closed convex set in R^3. Ann. of Math. (2) **43**, 739–741 (1942). [MF 7404]

An affirmative answer is given to the following query of Rademacher: Is it possible to circumscribe a cube about a given bounded closed convex set in three-space? The problem is solved by proving that for any real continuous function $f(P)$ defined on a two-sphere there exist points P_1, P_2, P_3, with $f(P_1)=f(P_2)=f(P_3)$ and such that the vectors OP_1, OP_2, OP_3 are orthogonal. The corresponding problem for n-dimensional space, $n\geqq 4$, is still unsolved.

N. Dunford (New Haven, Conn.).

Matumura, Sôzi. Über Flächen und Kurven. XXII. Einige Bemerkungen über die Theorie der Konvexkurven und -flächen. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. **28**, 259–297 (1940). [MF 7204]

Matumura, Sôzi. Über Flächen und Kurven. XXIII. Einige Bemerkungen über die Theorie der konvexen Flächen und Kurven. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. **28**, 323–378 (1940). [MF 7206]

Matumura, Sôzi. Über Flächen und Kurven. XXIV. Einige Bemerkungen über Eilinien und Eiflächen. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. **29**, 13–36 (1940). [MF 7208]

Matumura, Sôzi. Über Flächen und Kurven. XXV. Einige Bemerkungen über Eilinien und Eiflächen. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. **29**, 59–95 (1941). [MF 7210]

Santaló, L. A. Verallgemeinerung eines Satzes von T. Kubota über Eilinien. Tôhoku Math. J. **48**, 64–67 (1941). [MF 7138]

The author gives an integral geometric proof of the following theorem of Hermann Brunn [Ueber Kurven ohne Wendepunkte, Ackermann, Muenchen, 1889]: The curvature of two closed convex curves shall be positive and continuous everywhere. The minimum of the curvature of one of them shall be greater than the maximum of that of the other. Then the two curves intersect in no more than two points. *P. Scherk* (Bloomington, Ind.).

Santaló, L. A. Supplement to the note: A theorem on sets of parallelepipeds with parallel edges. Publ. Inst. Mat. Univ. Nac. Litoral **3**, 202–210 (1942). (Spanish. English summary) [MF 7139]

The paper quoted [Santaló, Publ. Inst. Mat. Univ. Nac. Litoral **2**, 49–60 (1940); these Rev. **2**, 261] contains the following theorem: If all the edges of a set of parallelograms are parallel to two directions and if any six parallelograms are intersected by suitable straight lines, then there is a line which meets all the parallelograms of the set. In the present paper the author quotes two examples by Frucht and Corominas ["problem" in Revista Union Mat. Argentina **7**, 134 (1941)] which show that the number six cannot be replaced by a smaller one. In both examples the parallelograms degenerate into segments not all of which are parallel. Analyzing them, the author proves: Given a set of segments, if all (all but one) of them are parallel, and if any 3 (any 5) are intersected by suitable straight lines, then there is a line which intersects all of them. *P. Scherk.*

Hadwiger, H. Gegenseitige Bedeckbarkeit zweier Eibereiche und Isoperimetrie. Vierteljschr. Naturforsch. Ges. Zürich **86**, 152–156 (1941). [MF 7372]

Let G and G_0 be two plane convex bodies, F and F_0 their areas, and L and L_0 their perimeters. In a previous paper [Comment. Math. Helv. **13**, 195–200 (1941); these Rev. **3**, 90] the author obtained sufficient conditions that G_0 be capable of covering G, that is, of being placed so that every point of G is an interior point of G_0. These conditions are $(L^2L_0^2-16\pi^2FF_0)^{\frac{1}{2}}<LL_0-4\pi F$ and $(LL_0^2-16\pi^2FF_0)^{\frac{1}{2}}<4\pi F_0-LL_0$, either of which is sufficient. In the present paper the author shows how these inequalities can be made to yield many of the known results which sharpen the isoperimetric inequality. He assumes two bodies G_0 and G_1, neither of which covers or is covered by G, and by manipulation of the above inequalities obtains several new ones involving G_0, G_1 and G. By specializing G_1 and G_0 to be the inscribed and circumscribed circles of G, many special inequalities result; for example, $L^2-4\pi F\geqq(L-2F/R)^2$, $L^2-4\pi F\geqq(L-2\pi r)^2$, $L^2-4\pi F\geqq\pi^2(R-r)$, and numbers of others. All these are known improvements on the isoperimetric inequality. Several new inequalities are obtained, which are in some cases stronger and in some cases weaker than the known inequalities. *J. W. Green* (Rochester, N. Y.).

Hadwiger, H. Bemerkungen über Gitter und Volumen. Mathematica, Timişoara **18**, 97–103 (1942). [MF 7425]

Let L be a unit lattice in an n-dimensional space and G an open set therein, called by the author a Jordan region. A closed Jordan region $\bar{G}$ consists of a Jordan region plus its limit points. A considerable amount is known about the result of translating G in the space; for example, if the volume of G is greater than 1, G can be translated until it contains not less than 2 lattice points. Also if the volume of $\bar{G}$ is 1, then $\bar{G}$ can be translated until it contains not less than 2 lattice points. In this paper the author studies similar problems in which "translation" is replaced by "rigid motion," or simply "motion," for briefness. The following two theorems are proved: (1) Any closed Jordan region of volume 1 can be suitably moved until it contains no lattice points. (2) A Jordan region of volume 1 can be moved until it contains not less than 2 lattice points; that is, the same result stated above for closed regions, except that movement replaces translation. The principal tool used in these investigations is the representation of the volume of G as the mean value of the number of lattice points contained in G, where the mean extends over the group of translations or rigid motions. *J. W. Green* (Rochester, N. Y.).

Stewart, B. M. A maximum problem. Amer. Math. Monthly **49**, 454–456 (1942). [MF 7149]

Main result: Given a finite set M of solids in space and a fixed point P. We consider the volumes V cut off by a variable plane p through P. Then a necessary condition for V to be a maximum is that P is the centroid of the area common to p and M, unless the area contains a plane face of a solid belonging to M. This result is closely related to a theorem of Bouguer and Dupin in hydrostatics [see Appell, "Traité de mécanique rationelle," vol. 3, 3rd ed., Gauthier-Villars, Paris, 1921, p. 198], according to which the planes cutting off equal volumes from a solid touch their envelope in the centroid of the area common to the plane and the solid. Indeed, both questions lead to the same variational equation $dV=0$. *F. John* (New York, N. Y.).

Differential Geometry

Comenetz, George. The limit of the ratio of arc to chord. Amer. J. Math. **64**, 695–713 (1942). [MF 7172]

Let C be an analytic curve in the three dimensional complex Euclidean space, which is not isotropic, but has a point O where the tangent t is isotropic. Denote by n and m, respectively, the orders of contact of C with t and the isotropic plane containing t. The limit R of the ratio of the arc of C to the chord of C at O is determined as follows: The corresponding problem for plane curves was solved by Kasner in "the ratio of the arc to the chord of an analytic curve need not approach unity" [Bull. Amer. Math. Soc. **20**, 524–531 (1914)]. If $m>2n$ then $R=\pm1$; if $m<2n$ then $R=2(m+1)^{\frac{1}{2}}/(m+2)$. If $m=2n$, then

$$R=\delta/[\delta^2-(m+2)^2/4(m+1)]^{\frac{1}{2}},$$

where δ is the derivative of the radius of curvature with respect to the arc length. The author discusses next how the invariants of a surface on which C lies partly determine m, n, δ. If the tangent plane π of S at O is not isotropic the relations are rather simple, but if π is isotropic they become involved and require the distinction of many cases which are completely discussed. *H. Busemann.*

Blumenthal, Leonard M. Note on an arc without tangents. Rep. Math. Colloquium (2) **4**, 3 pp. (1942). [MF 7573]

This paper defines a continuous curve in Hilbert space and proves that it has at no point either a progressive or regressive tangent of any kind. The continuous curve is in addition an arc, that is, the homeomorph of a segment. The methods used are those of distance geometry. Let S denote the metric space consisting of the numbers $0\leqq x\leqq1$

with distance defined as the positive square root of the Euclidean distance. It is shown, using known theorems, that S is congruently imbeddable in Hilbert space. Simple considerations show that S has at no point a progressive or regressive tangent. The functions $f_i(t)$, $i=1, 2, \cdots; 0 \leqq t \leqq 1$, which define the arc are not determined. *G. B. Price.*

Denk, Franz und Haupt, Otto. Über die Singularitäten reeller Bogen im R_n. J. Reine Angew. Math. **183**, 69–91 (1941). [MF 6839]

A space element in n-space is a set of $n+1$ (linear) subspaces of respective dimensions 0, 1, $\cdots$, n, such that for each $k>0$ the k-space of the set contains its $(k-1)$-space. An oriented space element is made up not of subspaces but of semisubspaces (a k-semispace is part of a k-space bounded by a $(k-1)$-space). The incidence relations between two space elements or between two oriented space elements can be described by means of a permutation of the numbers 1, 2, $\cdots$, n or a monomial substitution of them with the coefficients ± 1, respectively. If the pair of (oriented) space elements is ordered, the permutation (substitution) is uniquely determined.

An arc $\mathfrak{B}$ of finite order intersects each $(n-1)$-space in a finite number of points. If this number has a maximum, the maximum is called the order of $\mathfrak{B}$. The order of a point on $\mathfrak{B}$ is defined as the minimum of the orders of the neighborhoods of the point on $\mathfrak{B}$. The end-point of an arc of finite order is known to possess an osculating space element and an osculating oriented space element. Let the point P bisect the arc into two arcs of finite order. Thus P possesses a right and a left osculating oriented space element, which together determine a monomial substitution. Since this substitution determines the incidence relations between the two osculating oriented space elements, it can be said to describe the behavior of $\mathfrak{B}$ near P. The authors develop a rule for computing lower bounds for the order of P by means of this substitution and evaluate them in some cases, especially in the case of differentiable points, that is, points whose two osculating space elements are the same. Without any additional assumptions, the same lower bounds for the order of differentiable points have previously been obtained by the reviewer [Časopis Pest. Mat. Fys. **66**, 165–171 (1937)]. *P. Scherk* (Bloomington, Ind.).

Turrière, E. Sur des courbes gauches. Enseignement Math. **38**, 242–249 (1942). [MF 7301]

The author studies the space curves (C) whose osculating plane at a point M is parallel to OM', where M' and M are symmetric with respect to Oxy. The differential equation of these curves is found in the paper in terms of a general parameter and also in terms of several special parameters. Of these, σ, the area swept out by projection of the radius vector on the x-y plane, is the most extensively studied. If $x=x(\sigma)$ and $y=y(\sigma)$ are given, then z can be expressed as $z=Ax(i\sigma)+By(i\sigma)$. Several special cases are given. The author also finds the surfaces of which the asymptotic curves are the curves (C) to be the surface $z=f(x, y)$, where f is a homogeneous function of degree -1 in x and y.
J. W. Green (Rochester, N. Y.).

Abramescu, Nicolas. Sur les sections d'une surface par des plans menés par une tangente, ou par une droite passant par un point de la surface. Mathematica, Timişoara **18**, 146–150 (1942). [MF 7428]

Let C be a plane curve and O an ordinary point of C and t the tangent to C at O. Let M_1 and M_2 be two points of C. The line joining O to the midpoint of M_1M_2 approaches a line as M_1 and M_2 approach O, called by the author the harmonic normal D_n of C at O [cf. N. Abramescu, Sur les tangentes de Darboux d'une surface, Ann. Sci. Univ. Iassy **27**, 283–288 (1941)]. The characteristic point T_n of C at O is the pole of D_n with respect to the osculating conic of C at O. Let S be a surface and O be an ordinary point of S. Let t be a tangent to S at O, and C be the curve of section of S by a plane α through t. It is shown that the locus of the harmonic normal of C at O as α varies through the pencil with t as axis is a plane π. The locus of π as t varies through the pencil of tangent lines is a cone of order four with vertex at O. It is also shown that the characteristic point T_n of the curve of section C is the pole of the plane π with respect to the quadric of Moutard determined by either of the axes of the indicatrix of S at O. *V. G. Grove.*

Foster, Malcolm. Note on autopolar surfaces. Amer. Math. Monthly **49**, 589–595 (1942). [MF 7484]

In a previous paper [Bull. Amer. Math. Soc. **47**, 247–253 (1941); these Rev. **2**, 298] the author studied a particular contact transformation, namely, the polarity with respect to a parabola. In the paper under review the author extends the study to polarity with respect to the quadric $2z=x^2+y^2$. A surface is said to be autopolar if it is transformed into itself by the transformation. The surface $z=f(x, y)$ is autopolar if $f(x, y)$ satisfies the condition $f(x, y)+f(p, q)=px+qy$. He gives several properties of conjugate pairs, that is, pairs of points on the autopolar surface such that the polar of the first point is tangent to the surface at the second. In particular, every autopolar surface has at least one self-conjugate point. Finally several examples of autopolar surfaces are given. *V. G. Grove.*

Frucht, Roberto. A contribution to the elementary theory of surfaces. Revista Union Mat. Argentina **8**, 91–100 (1942). (Spanish) [MF 7482]

It is a result of Vakselj and H. W. Alexander that, while the mean curvature H of a surface, unlike the Gaussian curvature, is not a function of the first order fundamental quantities E, F, G and their derivatives, nevertheless H is not entirely independent of E, F, G, for H must in general satisfy two third order partial differential equations with coefficients which are functions of E, F, G and their derivatives. As a contribution toward the fuller understanding of the above result, the author obtains the more general theorem that, if any functions $L(u, v)$, $M(u, v)$, $N(u, v)$ satisfy two first order partial differential equations of a certain class (including the Codazzi equations as special case), then any functions $\Phi[u, v, L(u, v), M(u, v), N(u, v)]$ must in general satisfy two third order partial differential equations with coefficients which are functions of u, v, Φ and its derivatives. *E. F. Beckenbach* (Austin, Tex.).

Springer, C. E. Dual geodesics on a surface. Bull. Amer. Math. Soc. **48**, 901–906 (1942). [MF 7508]

The author defines the "ray point" corresponding to a point of a curve on a surface in Euclidean 3-space. In terms of these ray points, he also defines the system of "dual geodesics" of the surface and obtains their differential equation. Typical further results are: (1) If an asymptotic line is a dual geodesic, it is a straight line. (2) Isometric surfaces have the same equations of dual geodesics. (3) The locus of the ray points of all the geodesics through a fixed point on a surface is a cubic curve lying in the tangent plane to the surface at the fixed point. *A. Fialkow.*

Vest, M. L. A non-involutorial space transformation associated with a $Q_{1,n}$ congruence. Bull. Amer. Math. Soc. **48**, 767–773 (1942). [MF 7282]

The author discusses a non-involutorial transformation associated with a congruence of lines on a plane curve of order n having an $(n-1)$ point and a secant through that point. Consider a plane n-ic r, a line s meeting r at an $(n-1)$-point A, and two projective pencils of surfaces $|F_m| : s^{m-1}g_{2m-1}$ and $|F'_{m'}| : s^{m'-1}g_{2m'-1}$. A residual point is defined and the residual base curves of $|F|$ and $|F'|$, other than s, are denoted by g and g', respectively. Through a point on g' it is shown that there passes a unique line t' of the congruence which meets the associate surface of $|F|$ in a point $\bar{P}$. The points $\bar{P}$ generate a curve $\bar{g}$. Similarly, starting with a point on g, a curve $\bar{g}'$ can be generated. The author shows that r, s, g, g', $\bar{g}$, $\bar{g}'$ are fundamental curves of a non-involutorial transformation previously defined, and that the point A is a fundamental point of the second kind.

N. Coburn (Austin, Tex.).

***Lane, Ernest Preston. A Treatise on Projective Differential Geometry.** University of Chicago Press, Chicago, Ill., 1942. ix+466 pp. $6.00.

The author considers this treatise a sequel to his earlier volume [Projective Differential Geometry of Curves and Surfaces, University of Chicago Press, Chicago, Ill., 1932]. In the decade since the appearance of this earlier work, important new results have been obtained and new methods devised. The aim of the present work is to give a connected exposition of the theory to date. A treatise as extensive as this one, and as ambitious to report a full decade of important new work and to tie it in with all projective differential geometry since its inception, can hardly be expected to escape criticism on the score of topics omitted or included. Some of the material found in the earlier volume has been omitted, as, for example, the metric applications. On the other hand there has been considerable expansion of the older topics. For example, the work of Wilczynski on curves from the point of view of the theory of linear differential equations receives considerable attention, the previous volume giving scant attention to this method. On the other hand, the Fubini method by the use of differential forms is omitted. No mention is made of the study of surfaces at a point by plane sections through the point, nor of the neighborhoods of singular points of surfaces. Some attention is paid to certain types of singular points on curves. Approximately three times as much space is devoted to the theory of curves as was devoted to that subject in the earlier volume. The discussion of surfaces has also been extended about three times. More attention is paid to the fundamental background of the theory of surfaces, especially that part having to do with defining differential equations. The material on conjugate nets has been about doubled. In particular, the topic of conjugate nets in hyperspace is given in more detail, and that of conjugate nets in ordinary space is much expanded. The previous volume devoted one section to plane nets. The present volume devotes a twenty page chapter to the topic. Considerable factual material is presented at appropriate places as supplementary theorems. These theorems are not proved. In the earlier volume this was accomplished by giving the theorems as exercises. On the other hand, the textual material is more completely treated. Although in a treatise it is no doubt quite fitting to omit the excellent introductory sections, their omission represents a distinct loss. Much of the previously unpublished work of the author appears here in print for the first time.

The contents of the chapters of the treatise may be outlined as follows. I. Curves in hyperspace. Complete treatment by the method of Wilczynski by differential equations, and by the methods of power series expansions. II. Plane curves. The work of Halphen and the contributions of the author, Sannia and others are discussed. In particular, neighborhoods of inflexion points and sextactic points are considered. Relations between the methods of Wilczynski by defining differential equations and by power series are derived. The work of Fubini, Čech and Bompiani on the question of contact of two curves is covered. III. Space curves. The contributions of Halphen, Bompiani, Fubini, Čech, Palozzi and Stouffer on principal points, lines and planes are given due attention. As in the previous chapter, relations between the methods of differential equations and by power series are derived. Canonical forms of the power series and of the defining differential equation are given with the consequent geometrical theorems arising therefrom. IV. Surfaces. The simpler and more general portions of the theory are discussed. Conditions that a surface sustain a conjugate net, or an asymptotic curve or a net of such curves are derived. V, VI. Surfaces in ordinary space. Considerable attention is given to the various canonical forms of the defining differential equations. The power series are given to higher powers, with the consequent increase in the body of theorems. Considerable attention is paid to the canonical lines derived by various geometers. Mention is made of the work of Su, Green, Mori, Segre and the author on such topics as projectively minimal surfaces, projectively minimal curves, asymptotic curves belonging to linear complexes, projective curves, envelope of the quadric of Moutard, etc. VII. Conjugate nets in hyperspace. Sequences of Laplace are considered, and considerable new material appears apparently for the first time. Conditions that a net be conjugate or harmonic to a congruence are given as well as converse considerations. VIII. Conjugate nets in ordinary space. This thorough discussion of conjugate nets includes such standard topics as axis curves, ray curves, associate conjugate nets, isothermally conjugate nets, harmonic nets, Laplace transformations of nets, and newer topics such as canonical points, lines, planes, quadrics, principal joins, etc. Pencils of conjugate nets as developed by Wilczynski and the author occupy a more satisfactory space than devoted to them in the earlier volume. IX. Plane nets. Such topics as carry over from the general theory of surfaces and conjugate nets, or are analogous thereto, are discussed, as well as other topics such as osculating conics to the curves of the net, the congruentially associated net of Green, canonical points and lines, Koenig's theorem on the projection of the asymptotic net of a surface onto a plane, Green's contribution to that topic, and invariants under the transformation of Laplace. X. Transformations of surfaces. The transformations of Levy, of Ribaucour and the fundamental transformation of Jonas and Eisenhart are considered. The work of Bompiani on the induced transformation between bundles of osculating curves on two surfaces is discussed. The correspondence between the lines of a ruled space of three dimensions and the points of a hyperquadric in a linear space of five dimensions is used to study surfaces in three dimensions. W-congruences receive some attention. Some of the more recent contributions of Fubini [Ann. of Math. (2) **41**, 356–364, 620–638 (1940); these Rev. **1**, 271; **2**, 160] could have been mentioned. XI.

Surfaces and varieties in hyperspace. This concluding chapter is devoted to the work of the author, Bompiani, Tzitzéica, Stouffer, Mendel, Segre, Beenken, Bortolotti, Bertini and Cook on such topics as quasi-asymptotic curves, hyperplane sections of a surface, surfaces F, the surface of Veronese and varieties as loci of families of linear spaces. The volume ends with a bibliography of the works referred to in the text. *V. G. Grove* (East Lansing, Mich.).

Sprague, Atherton H. A differential geometry problem using tensor analysis. Bull. Amer. Math. Soc. **48**, 747–751 (1942). [MF 7278]

A conjugate net on a surface is called, following Tzitzéica [C. R. Acad. Sci. Paris **152**, 1077–1079 (1911)], an R net if the tangents to both families of curves of this net form W-congruences. A W-congruence is a congruence for which the asymptotic lines on the two focal surfaces correspond. In this paper we find the conditions that a given conjugate net on a surface be an R net. These conditions are presented in tensor form, and involve the fundamental tensors of the focal surfaces of the W-congruence. The relation to certain formulas given by Eisenhart [Differential Geometry, Boston, 1909, p. 115; Transformations of Surfaces, Princeton, 1923, p. 106] is established. *D. J. Struik.*

Choudhury, A. C. On the 4-web of the projective lines of curvature. Bull. Calcutta Math. Soc. **34**, 1–15 (1942). [MF 7163]

Fundamental to the study of projective differential geometry of curved surfaces are the differential equations of the surface in point coordinates

$$x_{,rs}=A^k_{rs}x_{,k}+P_{rs}x+G_{rs}y, \quad y_{,i}=B_i{}^r x_{,r}+L_i x,$$

and dually in plane coordinates

$$u_{,rs}=-A^k_{rs}u_{,k}+Q_{rs}u+G_{rs}v, \quad v_{,i}=P_i{}^r v_{,r}+M_i u,$$

and the forms

$$F_2=G_{rs}du^r du^s, \quad F_3=A_{rst}du^r du^s du^t,$$
$$P=P_{rs}du^r du^s, \quad Q=Q_{rs}du^r du^s.$$

The projective lines of curvature and the dual projective lines of curvature are given by the respective differential equations

$$B_i{}^r E_{qr}du^i du^q=0, \quad P_i{}^r E_{qr}du^i du^q=0,$$

wherein $E_{qr}+E_{rq}=0$. The lines of curvature fail to form a net if and only if the coefficients of the forms F_2 and Q are proportional. In this case the projective normals pass through a point. If the dual projective lines of curvature fail to form a net, then the coefficients of the forms F_2 and P are proportional. In that case the projective normal plane is a fixed plane. If the dual lines of curvature coincide with the curves $Q=0$, then the lines of curvature coincide with the curves of $P=0$. Under the assumption that the two nets of lines of curvature form a hexagonal 4-web, a necessary and sufficient condition that these nets be diagonal nets is that the dual lines of curvature be isothermally conjugate and coincide with $Q=0$. Moreover, if the two sets of lines of curvature form diagonal nets, the lines of curvature belong to the union curves of the projective normal congruence, and the dual lines of curvature belong to the dual union curves. *V. G. Grove* (East Lansing, Mich.).

Coburn, N. Frenet formulas for curves in unitary space. J. Math. Phys. Mass. Inst. Tech. **21**, 10–18 (1942). [MF 6775]

A set of Frenet formulas is given for a curve in an n-dimensional unitary space $\zeta^k=\zeta^k(t)$, $\zeta^{k^*}=\zeta^{k^*}(t)$, if t is real and ζ^k, ζ^{k^*} are conjugate. There are $3n-2$ curvatures, of which $2n-1$ are independent. Such a curve is uniquely determined if an initial point, or initial orientation of the n normals at that point, and $(2n-1)$ independent curvatures as functions of the arc length are given. Some properties of geodesics in a unitary space with semi-symmetric connection $S_{\lambda\cdot}^{\cdot\cdot k}=A^k_{[\nu}p_{\lambda]}$, (conj.) follow. *D. J. Struik.*

Coburn, N. Semi-analytic unitary subspaces of unitary space. Amer. J. Math. **64**, 714–724 (1942). [MF 7173]

A discussion of semi-analytic unitary subspaces S_m in an n-dimensional unitary space K_n. The analytic group is a subgroup of the group of S_m which in turn is a subgroup of the semi-analytic group of an Hermitian X_m. The metric tensor of S_m is discussed and it is shown that, if a coordinate system exists in S_m for which the ranks of certain matrices formed from the metric tensor of S_m are m, then S_m is a unitary Euclidean space. *T. Y. Thomas.*

Ghosh, N. N. The tortuosity of submanifolds of a variety. Bull. Calcutta Math. Soc. **33**, 187–195 (1941). [MF 7162]

This may be considered as a continuation of a recent paper [Bull. Calcutta Math. Soc. **32**, 51–60 (1940); these Rev. **3**, 191]. Using his notations based on "scalar determinants," the author amplifies some results of W. Mayer and gives a new derivation of a formula due to A. R. Forsyth concerning the relation between geodesics on the variety and those on the submanifold. *G. Y. Rainich.*

Davies, E. T. The first and second variations of the volume integral in Riemannian space. Quart. J. Math., Oxford Ser. **13**, 58–64 (1942). [MF 7450]

As the author points out, the main object of this note is to obtain the analogues of Euler's equations for a stationary V_m immersed in a Riemannian V_n and passing through a fixed boundary. These equations were first obtained by Bortolotti [Giorn. Mat. Battaglini **66**, 153–191 (1928)], but in this note they are derived from the second variation of the volume integral. Some well-known theorems about the first variation are also obtained. *M. S. Knebelman.*

MECHANICS

Birkhoff, G. D. Note on the law of the parallelogram of forces. Rice Inst. Pamphlet **28**, 46–50 (1941). [MF 8033]

This is an appendix to the author's lecture on "The Principle of Sufficient Reason" [Rice Inst. Pamphlet **28**, 24–50 (1941); these Rev. **3**, 132]. In this lecture, the author stressed some important applications of this principle to physics. As an example he mentioned that, using this principle, the familiar parallelogram law can be demonstrated. The appendix under review gives a strict proof of this assertion by deducing the parallelogram law from the following three axioms: (i) Collinear forces at a point O have a resultant in the same line; its magnitude is the sum of the magnitudes of the components. (ii) Any two forces have a unique resultant. The addition is associative and commutative and the resultant varies continuously with the

components; (iii) (Principle of Sufficient Reason). The resultant is independent of the choice of axes of reference and of the unit force. *W. Feller* (Providence, R. I.).

Shor, J. B. On the theorem of Roberts-Tchebyshev. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 323–324 (1941). (Russian. English summary) [MF 7717]

A simple proof of the theorem of Roberts-Tschebyshev on linkages is given by means of complex numbers. *J. D. Tamarkin* (Providence, R. I.).

Shor, J. B. On determination of accelerations in spatial mechanisms. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 491–494 (1941). (Russian. English summary) [MF 7730]

Nudelman, J. L. On the theory of stability of ideal frames. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 89–90 (1942). (Russian. English summary) [MF 7738]

Jackson, Dunham. The instantaneous motion of a rigid body. Amer. Math. Monthly **49**, 661–667 (1942). [MF 7816]

Dynamics

Birkhoff, G. D. The problem of the billiard ball and its significance in modern dynamics. Revista Ci., Lima **44**, 244–245 (1942). (Spanish) [MF 8038]

Birkhoff, G. D. On the dynamic stability. Revista Ci., Lima **44**, 248–249 (1942). (Spanish) [MF 8039]

Birkhoff, G. D. The ergodic theorems and their importance in statistical mechanics. Revista Ci., Lima **44**, 251 (1942). (Spanish) [MF 8040]

Birkhoff, G. D. The problem of n bodies. Revista Ci., Lima **44**, 252–253 (1942). (Spanish) [MF 8041]

Summaries (prepared from the author's notes) of lectures given by the author at the Universidad Nacional de San Marcos in Lima, April and May 1942.

Weinstein, Alexander. The spherical pendulum and complex integration. Amer. Math. Monthly **49**, 521–523 (1942). [MF 7343]

In any motion of a spherical pendulum the increment Φ of the azimuth corresponding to passage from the lowest level to the highest level is known to satisfy the two inequalities $\frac{1}{2}\pi<\Phi<\pi$, first proved by Puiseux and Halphen, respectively. Here Φ is given by a certain elliptic integral of the third kind, a discussion of which by analytic-function-theoretic methods gives simple and elegant proofs for both of these inequalities. The method also automatically shows (among other things) that the difference between Φ and $\frac{1}{2}\pi$ can be made arbitrarily small by choosing a motion which approximates sufficiently closely that of a simple pendulum with finite oscillations. It is thought that this result is new. *D. C. Lewis* (Durham, N. H.).

Zadunaisky, Pedro E. On problem no. 6 (Vol. 1, p. 34). Study on the Foucault pendulum. Math. Notae **2**, 73–94 (1942). (Spanish) [MF 7253]

This is a discussion of the trajectories described by a mass m under the action of an attractive force f which passes through a fixed point O and is proportional to the distance x of m from O. The mass is constrained to a motion in a plane π which rotates with a constant angular velocity ω about a vertical axis through O. Elementary considerations give the equations of motion in the plane π as sine functions of the time t (if $c\neq\omega^2$, where $c=f/mx$). If $\rho=(1-\omega^2/c)^{\frac{1}{2}}$ is irrational the motion is quasi-periodic; the trajectory covers the area of a rectangle with a density which tends to ∞ as $t\to\infty$. The proof is carried out by expanding ρ in continuous fractions. In a second part of the paper it is proved that under the assumption of the usual approximations the bob of Foucault's pendulum describes a variable ellipse. The result is known for this and for the more general type of pendulum of Gauss and Kamerlingh-Onnes [cf. Handbuch der Physik, vol. 5, Springer, Berlin, 1927, pp. 338, 449–452]. *I. Opatowski* (Chicago, Ill.).

Kimball, W. S. The special form of Coriolis' compound centripetal acceleration due to the rotating earth. J. Franklin Inst. **234**, 453–472 (1942). [MF 7374]

The author regards Coriolis' normal acceleration of a particle moving on the surface of the earth as consisting of two terms, one (and only one) of which is dependent on tangential acceleration. As Coriolis' acceleration is ordinarily considered to be a vector product of the velocity by the angular velocity of the reference frame, it is impossible for the reviewer to comprehend how it could depend on tangential acceleration. Perhaps the author really has in mind a force of smooth constraint. However this may be, he discusses motion in a uniform field of force. *D. C. Lewis.*

Seyfarth, Hellmut H. G. Nomographische Verfahren zur Berechnung und Konstruktion von Kreiselkörpern. Der günstigste Kreiselkörper, für den Material, Trägheitsmomentenverhältnis und Durchmesser vorgeschrieben sind. Luftfahrtforschung **19**, 236–239 (1942). [MF 7440]

Voznesensky, I. N. Principles and schemes of automatic regulation. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 101–110 (1942). (Russian) [MF 7741]

Nicolai, E. L. G. V. Tchipanov's paper "Theory and methods of calculating automatic regulators" printed in "Automatika and Telemechanika." J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 111–114 (1942). (Russian) [MF 7742]

Relativity, Astronomy

Birkhoff, G. D. The mathematical concept of time and gravitation. Revista Ci., Lima **44**, 253–257 (1942). (Spanish) [MF 8042]

The author has developed a New Theory of Relativity, a complete presentation of which has not yet appeared in print. The present paper is a summary of a lecture given May 15, 1942 at the Universidad Nacional Mayor de San Marcos, Lima, in which the new theory was first made public. The summary was prepared from the author's notes (but not by the author) and contains all formulas necessary. A complete outline of the theory is to appear soon.

Houstoun, R. A. Note on Einstein's theory of gravitation. Philos. Mag. (7) **33**, 899–903 (1942). [MF 7790]

The author replaces the usual relativistic assumptions on the path of a free particle and the path of light by the principle of least action and Fermat's principle. He retains, however, the principle of equivalence and the assumption that gravitational matter affects space and time. The latter assumption introduces a slight aelotropy into the element of optical length and the correction factors are obtained by means of the Lorentz-Fitzgerald contraction. In this way the author obtains the same results as those predicted by the three crucial tests of relativity. *M. Wyman.*

Berenda, Carlton W. The problem of the rotating disk. Phys. Rev. (2) **62**, 280–290 (1942). [MF 7197]

In part I of this paper previous treatments of the problem of the rotating disk are looked upon from the following standpoint: The points on a rotating disk are assigned coordinates x, y, z and t in terms of which the space-time element is

$$ds^2=[1-c^{-2}\omega^2(x^2+y^2)]dt^2 -c^{-2}[-2\omega(ydx-xdy)dt+dx^2+dy^2+dz^2],$$

where ω is the angular velocity of the disk as measured by a Galilean observer. Proper distances are taken as those measured perpendicularly (in a four dimensional sense) to the world lines of points at rest with respect to the disk (that is, those for which $dx^a/ds=0, a=1, 2, 3$). The spatial geometry of the disk is found to be that given by the line element

$$dl^2=dr^2+(1-c^{-2}r^2\omega^2)^{-1}r^2d\theta^2.$$

The paper concludes with a discussion of the behavior of clocks on a rotating disk and a possible application of the theory. *A. H. Taub* (Princeton, N. J.).

Martin, Monroe H. The restricted problem of three bodies. Trans. Amer. Math. Soc. **52**, 522–538 (1942). [MF 7459]

The restricted problem of three bodies considered here is that of the motion in a plane of an infinitesimal mass under the attraction of two finite masses moving in the same plane in a general solution of the two-body problem. F. R. Moulton has considered this problem in the case when the finite masses move in ellipses and the infinitesimal mass is free to move in space [Periodic Orbits, Carnegie Inst. of Wash. Publ., 1920, pp. 217–284]. The present paper studies the general two-dimensional problem and obtains detailed results for the parabolic case. It is shown that the differential equations can be written in a quasi-Lagrangian form:

$$\frac{d}{dt}\frac{\partial L}{\partial q_i'}-\frac{\partial L}{\partial q_i}+k(t)\frac{\partial L}{\partial q_i'}=0.$$

A number of properties of Lagrangian systems are shown to be capable of generalization to such systems, namely: the canonical form of the equations, the Hamilton-Jacobi partial differential equation, the energy integral (under the assumption that the system is "quasi-conservative"), the principle of least action, properties of the characteristic exponents at an equilibrium point. [Some of the results were obtained for a special case of these systems by Elliott [Ann. Sci. École Norm. Sup. (3) **10** (1893)].] The results obtained are then applied to the restricted three-body problem, especially to the determination of stable motions, motions which are bounded and bounded away from singularities (that is, collisions). In the parabolic case no periodic motions are possible, and any stable motion must be asymptotic to one of five libration points for $t\to\pm\infty$. In the case when the parabola degenerates into a straight line more precise results are found, including a proof of the existence of two stable motions when the finite masses are equal. *W. Kaplan* (Ann Arbor, Mich.).

Banerji, A. C. The instability of radial oscillations of a variable star and the origin of the solar system. Proc. Nat. Inst. Sci. India **8**, 173–197 (1942). [MF 7217]

The radial adiabatic oscillations of a variable star symmetrical about the centre are considered, the squares of the amplitude and of associated quantities being retained in the equations. It is proved that no finite radial oscillations exist either when the density of the star is uniform or when it decreases outwards as an inverse power of the distance from the centre (excluding the inverse powers 1 and 3) except in a small finite core of constant density. Reference is made to an unpublished investigation by a research student of the author [H. K. Sen] who proves that, for other density distributions, no finite radial oscillations exist even if the square of the amplitude is neglected. The theory is applied to the origin of the solar system. It is suggested that the Sun was originally a pulsating Cepheid variable with one or other of the permissible density-laws, the pulsations being small so that the squares of their amplitudes were negligible. The nearby passage of another star would disrupt the pulsations and cause matter to be ejected which would then form into the planets. The necessary energy for this process would be available. As an alternative hypothesis, both the Sun and its planets could be formed by the disruption of a more massive Cepheid due to the nearby passage of another star. The necessary energy conditions would again be satisfied. *G. C. McVittie* (London).

Giraud, Georges. Sur une figure d'équilibre relatif d'une masse tournante constituée par un liquide et par un noyau solide immergé. Bull. Sci. Math. (2) **64**, 268–298 (1940). [MF 6789]

Consider a homogeneous solid of mass M_1, density ρ_1 and a homogeneous liquid of mass M_2, density ρ_2. Suppose $M=M_1+M_2$ is the total mass, and ρ the mean density, that is, $M/\rho=M_1/\rho_1+M_2/\rho_2$. Let E be the surface of an ellipsoid of rotating equilibrium for a homogeneous fluid of mass M and density ρ rotating with angular velocity ω. Let the solid mass M_1 take the form and position of an ellipsoid E_1 homofocal to the surface E. Fill in the rest of the ellipsoid with the liquid mass M_2 and do not change the angular velocity. The author proves that, if $\rho_1>\rho_2$ and if E is an ellipsoid of revolution such that the eccentricity of its meridional section is small enough, then the total energy of the rotating system is a relative minimum.
B. Friedman (Chicago, Ill.).

Fessenkoff, V. Théorie dynamique de la lumière zodiacale. Astr. J. Soviet Union [Astr. Zhurnal] **19**, 28–49 (1942). (Russian. French summary) [MF 7763]

* According to the final volume index the correct quotation is: 19, no. 4, pp. 28–39 (1942). The pagination of this particular issue starts with p. 1.

Hydrodynamics, Aerodynamics, Acoustics

Astrov, G. G. On the calculation of the kinetic energy of fluids. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal. Prikl. Mat. Mech.] **5**, 319–322 (1941). (Russian. English summary) [MF 7716]

Hantzsche, W. and Wendt, H. The compressible potential flow past elliptic symmetrical cylinders at zero angle of attack and with no circulation. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1030, 1–16 (1942). (2 plates) [MF 7410]

Translation of a paper in Luftfahrtforschung **18**, 311–316 (1941); these Rev. **3**, 283.

Ballabh, Ram. On superposability. J. Indian Math. Soc. (N.S.) **6**, 33–40 (1942). [MF 7233]

The author is interested in unsteady fluid motions in which the components of vorticity (ξ, η, ζ) are connected with the components of velocity by equations of type $\xi=\lambda u$, $\eta=\lambda v$, $\zeta=\lambda w$, where λ is generally a function of the coordinates. A case in which λ is constant is furnished by

$$u=0, \quad \lambda v=-e^{-\nu\lambda^2 t}[\alpha J_0'(\lambda r)+\beta Y_0'(\lambda r)], \quad \lambda w=e^{-\nu\lambda^2 t}[\alpha J_0(\lambda r)+\beta Y_0(\lambda r)],$$

where r is the distance from the axis of cylindrical coordinates. It is shown that this type of solution is applicable only when the fluid extends to infinity. The motion decays exponentially with time. The possibility of λ being a function of type $f(s, t)$ is then considered, the case in which $s=r$ being discussed in some detail. A treatment is then given in spherical polar coordinates of the three-dimensional problem. *H. Bateman* (Pasadena, Calif.).

Busemann, A. Die achsensymmetrische kegelige Überschallströmung. Luftfahrtforschung **19**, 137–144 (1942). [MF 7436]

Consider an irrotational stationary flow of a perfect fluid with velocity potential ϕ. Then it is well known that

$$(1) \quad \sum\varphi_{ij}\left(\delta_{ij}-\frac{u_iu_j}{a^2}\right)=0, \qquad \varphi_{ij}=\frac{\partial^2\varphi}{\partial x^i\partial x^j}, \qquad \vec{u}=(u_1, u_2, u_3),$$

is the velocity, $\delta_{ij}=1$ or 0 and a, the velocity of sound, depends on $\sum u_i^2$. The Legendre transformation $\chi(u_1, u_2, u_3)=\sum u_ix^i-\phi$ yields $\partial\chi/\partial u_i=x^i$. Conical flows are characterized by invariance of the stream lines under $\vec{r}=k\vec{r}$. Thus along a ray, $\vec{u}$ and the pressure and density are constant. Hence $\phi=\sum u_ix^i+C$ and therefore (2) $\chi=C'$. Since $d\vec{u}=0$ for $d\vec{r}=kd\vec{r}$, (3) φ_{i1}, $i=1, 2, 3$, may be expressed in terms of φ_{rs}, $r, s=1, 2$, and, using (2), $u^3=u_3(u_1, u_2)$. Thus the problem is essentially two-dimensional. On $x^3=C''$ we have (4) $du_r=\sum\phi_{rs}dx^s$, $r, s=1, 2$. On using (3) and (4), assuming the Hessian of u^3 is not zero, (1) becomes a partial differential equation (referred to as (5)) in $u_3(u_1, u_2)$. The writer treats the special case of axial symmetry, for which $\chi=C$ is a surface of revolution about the u_1 axis. Let w be the tangential velocity along a meridian, then $(u_3)^2=(w(u_1))^2-(u_2)^2$. Then $w(u_1)=u_3(u_1, 0)=U$ and (5) reduces to

$$(6) \qquad UU''=1+(U')^2-\left(\frac{u_2+UU'}{a}\right)^2.$$

This plays a central role in the paper. Various approximations (mostly truncated series), graphs, numerical computations and some semi-intuitive arguments are used to characterize the sub- and supersonic cases. These indicate for instance, that there are essentially just two kinds of such flows and that a "density shock" is essential for their existence. Since the vertex, and in some cases the entire x' axis, are singular elements, it seems to the reviewer that more powerful analysis would be of interest; a natural first suggestion is the use of asymptotic solutions (in a parameter) on the plan of modern boundary layer treatments in hydrodynamics and elasticity. *D. G. Bourgin.*

Betz, A. Verlauf der Strömungsgeschwindigkeit in der Nähe einer Wand bei unstetiger Änderung der Krümmung. Luftfahrtforschung **19**, 129–131 (1942). [MF 7435]

In the earlier paper [Luftfahrtforschung **17**, 189–195 (1940); these Rev. **2**, 168] von Koppenfels investigated the behavior of an ideal fluid flowing past a wall having discontinuous curvature at a point. He investigated conditions at the wall and found that $d|w|/d\sigma$ becomes infinite at the point of discontinuity, where $w=u+iv$ is the velocity vector and σ is the distance along the wall, and concluded that separation of the boundary layer at such a point might be expected in the case of a real fluid. The present author repeats Koppenfels' calculation by a somewhat simpler method, obtaining an expression for $\ln|w|$ that can be evaluated both at the wall and in the fluid near the singularity. As an example he considers a case where the wall consists of two parallel straight lines displaced relative to one another by a small distance δ and joined by an S-shaped curve consisting of two circular arcs of radius R. The wall has discontinuous curvature at three points a distance s apart; $\delta\leqq s^2/R$. The numerical calculations for this case show that, although $d|w|/d\sigma$ becomes infinite at the wall, the maximum value of this derivative diminishes rapidly as the distance from the wall is increased. The author concludes that the infinite derivative at the wall probably does not influence the boundary layer. He points out that, on the other hand, his results show considerable deviation of the magnitude of the velocity near the singularity from the undisturbed value (somewhat over ten per cent in the case $\delta/s\leqq s/R=0.1$), and that these deviations do not diminish rapidly as the distance from the wall is increased. The accompanying changes in pressure may influence the boundary layer profoundly. *W. R. Sears.*

Thomas, T. Y. Qualitative analysis of the flow of fluids in pipes. Amer. J. Math. **64**, 754–767 (1942). [MF 7177]

The paper deals with the stability of motion of a viscous incompressible fluid in a straight pipe of circular section. The exact Navier-Stokes equations are used throughout. The disturbance is assumed to be spatially periodic along the pipe, and the method is based on consideration of the kinetic energy of the disturbance in a periodicity-cell. The author is able to establish stability under the assumption that at all times the discharge (or flux) is greater for the disturbed than for the undisturbed motion, the pressure-drop being the same in the two cases. Conversely, it follows that in established turbulence the discharge is less than in steady flow. Finally, the author assumes a disturbance of the usual type with an exponential factor $\exp(\sigma t+i\lambda z)$, and arrives at the (physically) surprizing result that the steady flow is always stable with respect to such disturbances. It must be pointed out, however, that this conclusion rests on the assumption that the exact nonlinear equations of disturbance possess solutions of this exponential type.

J. L. Synge (Toronto, Ont.).

***A Study of Progressive Oscillatory Waves in Water.** Technical Report, no. 1. Office of the Chief of Engineers, Beach Erosion Board. U. S. Government Printing Office, Washington, D. C., 1941. vi+39 pp. $.50.

This paper deals with a comparison of the known theoretical results of progressive oscillatory waves in water [see references in the following review] and experimental results.

A. E. Heins (Cambridge, Mass.).

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***A Summary of the Theory of Oscillatory Waves.** Technical Report, no. 2. Office of the Chief of Engineers, Beach Erosion Board. U. S. Government Printing Office, Washington, D. C., 1942. vi+43 pp. $.10.

This paper deals with a summary of the theoretical treatments of oscillatory wave motion on a free water surface [Rayleigh, Philos. Mag. (6) **21**, 177–195 (1911); ibid. (6) **33**, 381–389 (1917); Scientific Papers, vol. 6, (1920); Levi-Civita, Math. Ann. **93**, 264–314 (1925); Stokes, Trans. Cambridge Philos. Soc. **8**, 441–455 (1849)] and is limited to the type of wave motion encountered in shore protection problems. A discussion of the known experimental results is given. *A. E. Heins* (Cambridge, Mass.).

Bondi, H. On the generation of waves on shallow water by wind. Proc. Roy. Soc. London. Ser. A. **181**, 67–71 (1942). [MF 7257]

H. Jeffreys showed in two fundamental publications [Proc. Roy. Soc. London. Ser. A. **107**, 189–206 (1925); **110**, 241–247 (1926)] that the generation of water waves in deep and in moderately shallow water by wind can be well described if the motion of the water is considered to be irrotational in first approximation, but the vorticity in the air (or the "discontinuous" motion of the air which approximately represents this vorticity) is taken into account. The present author extends Jeffreys' theory to very shallow water down to a depth of about 0.5 cm. by taking into account the energy dissipation in the boundary layer at the bottom, while retaining the assumption of irrotational water movement near the water surface. The agreement of this theory with experimental facts seems to be satisfactory.

P. Neményi (Fort Collins, Colo.).

Jacob, Caius. Sur un problème au contour de la théorie des marées. Mathematica, Timişoara **18**, 151–158 (1942). [MF 7429]

Consider a simply connected domain Ω in the plane (x, y). The boundary of Ω is an analytic curve C. The boundary value problem is to determine a complex function $F(x, y) = U_1(x, y)+iU_2(x, y)$, where $U_1(x, y)$ and $U_2(x, y)$ are harmonic functions regular in Ω and have continuous first partial derivatives in $\Omega+C$, such that, at C,

$$\partial F/\partial n+iA\partial F/\partial s=g(s),$$

where $g(s)=g_1(s)+ig_2(s)$ is a complex function defined on C and satisfies a Hölder condition with respect to the parameter s; n is the interior normal. This problem is connected with the characteristic oscillation of a heavy liquid in a vessel of constant depth rotating uniformly around a vertical axis. In this case A is a real constant proportional to the angular velocity of rotation. It is shown that then the problem can be reduced to two Neumann problems.

The more general case of A as a function of s, satisfying a Hölder condition, is connected to the theory of ocean tide [G. Bertrand, Ann. Sci. École Norm. Sup. (3) **40**, 151–258 (1923); in particular, p. 234]. It is shown that, if $A^2\neq 1$, then $\int_C g(s)ds=0$ is the necessary and sufficient condition for the existence of a unique solution of the problem, except an additional constant. *H. S. Tsien* (Pasadena, Calif.).

Pavlov, A. T. Steady flow of ground water consisting of two strata of different densities. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 221–228 (1942). (Russian. English summary) [MF 7752]

Braginskaya, V. A. On the problem of filtration in anisotropic soil. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 229–240 (1942). (Russian. English summary) [MF 7753]

The article deals with the case in which the principal axes of filtration at every point, and the coefficients of filtration along them are constant. The ground is saturated with two fluids; the lower is of greater density and is at rest, the upper is in a state of motion. The motion is assumed to be steady. By means of the method given by P. J. Polubarinova-Kochina the characteristic function of flow and the configuration of the demarcation line between the two fluids are investigated. The problem is solved for the cases of a dam with sheet piling and a flood-bed.

Author's summary.

Numerov, S. N. Filtration without percolation and without infiltration or evaporation from the surfaces. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 75–86 (1942). (Russian. English summary) [MF 7736]

Polubarinova-Kotschina, P. J. Inflow of fluids to oil wells in a heterogeneous medium. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 42–46 (1942). [MF 7596]

The author considers two porous media of different permeability, each bearing a liquid of different viscosity. An elementary expression for the complex velocity potential and the rate of production is obtained in the case of two semi-infinite media separated by a plane and in the case in which one of the media is a circular cylinder and the other the remaining space. In both cases the pressure is assumed to be known on two isobars. The case in which one medium is a circular cylinder and the other a coaxial cylindrical ring is also considered. *I. Opatowski* (Chicago, Ill.).

Loytsansky, L. G. Resistance to motion through a liquid of a body surrounded by a layer of a fluid whose physical constants differ from those of the liquid. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 95–100 (1942). (Russian. English summary) [MF 7740]

Simonov, L. A. On three-dimensional flow in hydroturbines. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 471–488 (1941). (Russian. English summary) [MF 7728]

Tollmien, W. Ein Wirbelsatz für stationäre isoenergetische Gasströmungen. Luftfahrtforschung **19**, 145–147 (1942). [MF 7437]

The case considered is that of a solid body moving with uniform supersonic speed through an ideal gas. Forward of the shock wave the flow is isentropic; the entropy increases abruptly at the wave and then remains constant along each streamline behind the wave (neglecting friction and heat transfer). For such flows the author transforms the equations of motion into the form

$$(1)\qquad (\mathrm{rot}\ \mathbf{v})\times\mathbf{v}=(1/2\kappa)(V_m^2-\mathbf{v}^2)\ \mathrm{grad}\ \ln f,$$

where $\mathbf{v}$ is the velocity vector, κ is the ratio of specific heats and f denotes p/ρ^κ, where p is the pressure and ρ the density; $\frac{1}{2}V_m^2$ is the total energy per unit mass, defined by

$$\tfrac{1}{2}V_m^2=\tfrac{1}{2}\mathbf{v}^2+(\kappa/(\kappa-1))(p/\rho),$$

and has a constant value throughout the flow. Along any

streamline the function f assumes a constant value determined by the configuration of the shock wave.

In a two-dimensional case, if the stream function ψ is introduced (defined by $\rho u=\partial\psi/\partial y$ and $\rho v=-\partial\psi/\partial x$), equation (1) can be reduced to the form.

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{p}{\kappa-1}\frac{d(\ln f)}{d\psi}.$$

Equation (1) is a generalization of the theorem $(\text{rot } \mathbf{v})\times\mathbf{v}=0$ employed in wing theory for incompressible fluids. The author discusses the relationship of the new theorem to one of L. Crocco and to the classical vortex laws. *W. R. Sears.*

Loytzansky, L. G. On integral methods in the theory of the boundary layer. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 453–470 (1941). (Russian. English summary) [MF 7727]

The application of the well-known basic principle of mechanics, the principle of Jourdain, to problems of the theory of the boundary layer leads to an equation from which the equations of Kármán, Leibenson, and Golubev are derived as special cases. The given equation may be employed in other integral methods. The present work deals with the variation of the thickness of the boundary layer. A number of new approximate formulae for distribution of friction are derived from this procedure. These formulae are valuable in aerodynamics calculations. The method has been applied only to laminar layers; but it seems probable that it may be generalized to include turbulent layers as well. *Author's summary.*

Loytzansky, L. G. Approximate method for calculating the laminar boundary layer on the airfoil. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 227–232 (1942). [MF 7619]

The velocity profiles in the boundary layer are assumed to be of the form

$$u(x, y)/U(x)=1+a_1\zeta^n+a_2\zeta^{n+1}+a_3\zeta^{n+2},$$

where $U(x)=u(x, \delta)$ is the velocity at the free stream boundary of the layer $y=\delta(x)$, the variable ζ stands for $1-y/\delta$ and a_1, a_2, a_3, n are functions of x. The boundary conditions $u=0$, $\partial^2u/\partial y^2=-UU'/\nu$, $\partial^3u/\partial y^3=0$ allow one to express a_1, a_2, a_3 in terms of n and Pohlhausen's parameter $\lambda=U'\delta^2/\nu$. The momentum theorem applied to the boundary layer then furnishes a relation for the momentum thickness

$$\delta^*=\int_0^\delta\frac{u}{U}\left(1-\frac{u}{U}\right)dy,$$

which is integrated under the assumption of a linear relation between n and λ. The results are found to agree well with known solutions. *W. Prager* (Providence, R. I.).

Schmidt, Harry and Schröder, Kurt. Laminare Grenzschichten. Ein kritischer Literaturbericht. I. Teil. Grundlagen der Grenzschichttheorie. Luftfahrtforschung **19**, 65–97 (1942). [MF 7433]

This first part of a comprehensive report on the theory of laminar boundary layers deals with the basic conceptions and equations. The ground covered is best indicated by the following section headings: A. The general equations of motion of hydrodynamics: 1. The equation of continuity, the impulse theorem and the Navier-Stokes equations. 2. The equations of motion in orthogonal curvilinear coordinates. 3. Parallel curves and normals of a plane arc as coordinate lines. B. Introduction of the boundary layer equation: 1. The procedure of Prandtl and Blasius. 2. Complements and generalizations. 3. A systematic study of the neglections made in boundary layer theory. 4. A rigorous solution of the Navier-Stokes equations as an example. C. The assumptions of v. Kármán and Pohlhausen: 1. Kármán's stream function. 2. Pohlhausen's modification of Kármán's assumption. 3. Kármán's integral condition. D. The theory of v. Mises: 1. The fundamental equations. 2. The analogy with heat conduction. 3. The equations of a steady two-dimensional motion with respect to the stream lines and their orthogonal trajectories. E. Related ideas: 1. A remark of Jeffreys. 2. R. H. Smith's theory of the boundary layer based on a minimum principle. 3. Klose's conception of the boundary layer theory as part of a theory of fluids with vanishing viscosity. Throughout this report the authors have endeavored to make the arguments more rigorous than in the usual presentation of the theory. This remark applies particularly to section B, 3, where the following asymptotic relation between the thickness δ of the boundary layer and the Reynolds number R is established: $\delta=O(1/R)\neq o(1/R)$. *W. Prager* (Providence, R. I.).

Jaeckel, Karl. Zur Theorie der tragenden Linie im Instationären. Luftfahrtforschung **19**, 57–63 (1942). [MF 7432]

The author first sets up the basic integral equation for the velocity potential $\Phi(x, y, z, t)$ in the case of a lifting surface of small camber at a small angle of attack, lying nearly in the xy plane. The integration is to be carried out over both the wing surface and the discontinuity surface (wake) behind it. He also writes the analogous integral equation for the two-dimensional case. These equations are manipulated into several forms by introducing the conditions that the potential discontinuity vanishes at the boundaries and has the same values on the wing and in the wake at the trailing edge. For wings of high aspect ratio the author makes a simplifying approximation in the kernels, which is believed to cause only small inaccuracies. The resulting integral equation is written in such a form that the velocity component $\partial\Phi/\partial z(x, y, 0, t)$ is given by the sum of two terms: (1) the value calculated in a two-dimensional case having the circulation distributions on the wing and in the wake that actually exist at the spanwise station y, and (2) a correction term that involves integration over the entire wake. In the stationary case, if the variation of $\partial\Phi/\partial z$ across the chord at any value of y is neglected, this integral equation reduces to that of the well-known Prandtl first-order wing theory.

Returning to the nonstationary case, the author assumes slow periodic motion, and makes the usual assumption that the wake vortices travel downstream relative to the wing with the undisturbed velocity. He then proceeds to evaluate the correction term mentioned above, and finds that it consists of (a) the usual Prandtl correction, independent of the frequency, and (b) terms dependent upon the frequency that are shown to be small compared to the Prandtl correction and can therefore be neglected. He then takes the two-dimensional nonstationary values from an earlier paper [Luftfahrtforschung **16**, 135–138 (1939)] and obtains a final formula for the distribution of total circulation across the span. The nonstationary corrections vary along the span and involve both in-phase and out-of-phase components; the author proposes to employ mean values of the corrections and to neglect the out-of-phase components entirely. This implies the use of a "quasi-stationary" wing theory for such cases. *W. R. Sears* (Inglewood, Calif.).

Stewart, H. J. A simplified two-dimensional theory of thin airfoils. J. Aeronaut. Sci. **9**, 452–456 (1942). [MF 7265]

Consider two-dimensional irrotational flow of an incompressible fluid. The velocity components u and v satisfy the Cauchy-Riemann equations. The author presents the so-called "thin airfoil theory" in a new form by working directly with the complex velocity $w=u-iv$ without introducing a velocity potential; w is obviously an analytical function of $z=x+iy$. If w is determined the pressure distribution, lift and moment are found in the usual way by applying Bernoulli's equation and the Blasius relations. The author applies this method to the flat plate airfoil, the cambered airfoil of zero thickness and the cambered airfoil of small thickness. Finally two numerical examples are given: the first is a solution of the "direct" problem, the determination of pressure distribution, lift and moment for a parabolic airfoil; the second treats the inverse problem, that of finding the camber function for a given lift distribution (constant lift along the leading half, zero lift along the trailing half of the cord). *H. W. Liepmann.*

Vetchinkin, V. Review of papers by N. Joukovsky and S. Chaplygin which lay the basis of the wing theory. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 161–164 (1941). [MF 7705]

Gagut, A. F. Approximate calculation of the flutter of an aeroplane wing. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 115–122 (1942). (Russian. English summary) [MF 7743]

The work presents a means of calculating proper values of nonhomogeneous boundary problems in mechanics. This procedure is based on reducing the problem to integral equations. The nuclei of these equations are replaced by other approximate nuclei. *Author's summary.*

Millionščikov. M. On the theory of homogeneous isotropic turbulence. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 433–446 (1941). (Russian. English summary) [MF 7325]

The third moments of the components of the velocities in a homogeneous isotropic process of decay of turbulence are investigated. Differential equations connecting second, third and fourth moments are set up. These equations are solved assuming approximate second moments obtained in a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 231–235 (1939)] and assuming that the fourth moments have approximately the same relation to the second moments as in normal distributions. The author considers this approach more realistic than that of von Kármán and Howarth [Proc. Roy. Soc. London. Ser. A. **164**, 192–215 (1938)], who obtained the third moments from such differential equations by neglecting all fourth moments.

J. L. Doob (Washington, D. C.).

Obuhov, A. Über die Energieverteilung im Spektrum des Turbulenzstromes. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 453–466 (1941). (Russian. German summary) [MF 7326]

This paper contains a detailed exposition of the contents of an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 19–21 (1941); these Rev. **3**, 221]. *J. L. Doob.*

Obuchow, A. M. Ueber die Verteilung des Turbulenzmasstabes in Strömen mit beliebigem Querschnitt. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 209–220 (1942). (Russian. German summary) [MF 7751]

Frankl, F. Heat transfer in the turbulent boundary layer of a compressible gas at high speeds. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1032**, 1–8 (1942). [MF 7411]

Translation of Report no. 240 of the Central Aero-Hydrodynamical Institute, Moscow, 1935.

Ginsburg, V. L. On the dispersion of high frequency acoustic waves in liquids. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 8–13 (1942). [MF 7625]

Theory of Elasticity

Malkin, I. On a generalization of Kirchhoff's theory of transversal plate vibrations in the vibration problem of steam turbine disks. J. Franklin Inst. **234**, 355–369, 431–452 (1942). [MF 7373]

The problem of calculating the axial vibrations of steam turbine disks has been carried out in the past by the Rayleigh-Ritz method and the generalization of this method due to Stodola [Dampf- und Gas-Turbinen, Springer, Berlin, 1922]. Such results are valid provided these disks do not have long blades or heavy rims with two or more rows of blades. With the exception of these latter cases, this problem can be formulated mathematically as a calculus of variation problem [G. Kirchhoff, Vorlesungen über Mathematische Physik, vol. I, Teubner, Leipzig, 1897, p. 459] and assumes that the disk is a plane circular plate. Hence Kirchhoff's original formulation cannot be applied immediately to the problem of transverse vibrations of a certain class of turbine disks because this class of disks is essentially a plate of variable thickness and has blades at the rim. If the blades of the disk are under a certain length, the entire disk may be considered as participating in the vibration. If the blades exceed this length, the vibration phenomenon will occur mainly in the blades and only an outer ring of the disk will be involved in this vibration. On the basis of these remarks, the author generalizes Kirchhoff's original formulation of the disk problem. This problem can be finally formulated as a boundary value problem which involves the solution of two fourth order partial differential equations under boundary conditions. Each of these differential equations holds in different regions of the disk and they must be tied together at the common boundary of the region. Numerical examples are given. *A. E. Heins.*

Prescott, John. Elastic waves and vibrations of thin rods. Philos. Mag. (7) **33**, 703–754 (1942). [MF 7479]

In vector form the writer's analysis amounts to the following: Let $\vec{u}=(u^1, u^2, u^3)$ be the displacement and ϕ and $\vec{\psi}$ the scalar and vector potentials. Then (1) $V_D^2\nabla^2\varphi=\varphi_{tt}$ and (2) $V_s^2\nabla^2\vec{\psi}=\vec{\psi}_{tt}$ are the dilatational (D) and rotational (R) wave equations. The arguments of the paper are formal and involve induction from numerical computation for special cases. The first results are really well known—for instance, that plane D waves with $V=V_D$ propagate with no change

in form, that reflection at a boundary of either D or R waves gives rise to both D and R reflected waves, that the reflection and incidence angles need not be equal for R waves, etc. The main contribution is the study of the exact equation for propagation of transverse disturbances in a rod and the feasibility of an approximate equation for the phenomena. A new approximate equation is derived, namely

$$(3)\qquad \left(V_f^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\left(\frac{1}{V_s^2}W_{tt}-W_{xx}\right)=\frac{\alpha}{k^2}W_{tt},$$

where W is a certain mean value of u^3 and is expected to be close to the displacement of the rods' central axis. K is the gyration radius about this axis and α and V_f are empirical constants. (However, a somewhat similar equation occurs in work of Timoshenko). A rectangular rod of width $2b$, large in comparison with the depth, yields essentially a two-dimensional problem with z transverse and x longitudinal if $u^2=0$. Consider solutions of the form (4) $\phi=\varphi_1 e^{c(x-Vt)}$, (5) $\psi=\psi_1 e^{c(x-Vt)}$. Take (6) $c\varphi_1=J\sin chz$, (7) $c\psi_1=H\cos ckz$, where $h^2=1-(V/V_D)^2$, $k^2=1-(V/V_s)^2$. Write (8) $\vec{u}=\nabla\phi+\nabla x\vec{\psi}$. The boundary conditions in the stresses are (9) $T_{zz}|_{\pm b}=0$, (10) $T_{xz}|_{\pm b}=0$. Assume the customary linear stress strain relations. Then (9) and (10) yield, on elimination of T/H, the central relation

$$(11)\qquad \frac{\tan chb}{\tan ckb}=\frac{(1+k^2)^2}{4hk}.$$

The left side of (11) is even in cb. Hence (11) may be written (12) $(cb)^2=F(V^2)$. For each real V there are associated real or imaginary c values corresponding to periodic or damped (enhanced) waves. (Cf. (4) and (5).) (No reason is given for omitting complex V values and perhaps undue consideration is given the case $c>0$ for obviously the assumed approximations are now invalid.) Now (12) is obviously (infinitely) multiple valued and for the principal branch, $F(0)=0$, $c=i\infty$ for some $V=V'$, $0\leqq V'\leqq 1$. The other branches of (12) yield real c values only. For $V\sim V'$ and x,t fixed, (13) $T_{xz}\sim c'(e^{-c'kd}-e^{-c'kd})$, where c (real) $=ic$ and d is the distance from the nearest boundary. Thus the shear may be large in a thin skin bordering the boundary, a situation not comprehended by the usual approximate equations. However, by slight changes in (3), a fairly satisfactory fit for the principal branch of (12) is obtained though the skin stresses are not accurately portrayed. An interesting discussion indicates no energy is propagated with $V>V_D$ though surges occur, but there is no mention of classical group velocity arguments for somewhat analogous situations.

D. G. Bourgin (Urbana, Ill.).

Pipes, Louis A. The matrix theory of torsional oscillations. J. Appl. Phys. 13, 434–444 (1942).

The characteristic equations for linear rotational systems consisting of sets of rigid disks on a shaft, and the equations for a system of gears, are obtained by a chain process similar to that common in electric-transmission-network theory. [See for example T. R. Rosebrugh, Toronto Univ. School Engrg. Research. Bull. 1, 169–190 (1919)]. A system is treated as a chain of sections. Assuming that the output variables are linearly related to the input variables, passage is made through successive sections of the chain by linear algebraic transformations. Of course the process can be described in terms of matrices and their products. The treatment is given in these terms. *J. L. Barnes.*

Higgins, Thomas James. A comprehensive review of Saint-Venant's torsion problem. Amer. J. Phys. 10, 248–259 (1942). [MF 7651]

The paper gives a useful enumeration of the cross-sections for which exact solutions of B. de Saint Venant's torsion problem have been obtained. The bibliography contains 89 numbers. *W. Prager* (Providence, R. I.).

Sokolnikoff, Ivan S. Some new methods of solution of two-dimensional problems in elasticity. Bull. Amer. Math. Soc. 48, 539–555 (1942). [MF 7047]

It is well known that the two dimensional boundary problems of potential theory such as those of Dirichlet and Neumann are resolved by means of the theory of functions of a complex variable. In particular, if $U(x, y)$ is harmonic in a closed and simply connected region R and if it attains continuous values $f(s)$ on the boundary C of R, then a single analytic function $\phi(z)$ is to be found such that

$$\phi(z)+\phi(z)=2f(s)$$

on C, where U is the real part of ϕ. If R is a circle or if it is conformally representable upon the unit circle in the ζ plane by $z=\omega(\zeta)$, then $\phi(z)$ or its transform $\phi_1(z)=\phi[\omega(\zeta)]$ is determined by the integral formulation of Schwartz. In an analagous manner N. Muschelisvili and a group of Russian workers have reduced the biharmonic boundary value problem to the determination of two analytic functions of a single complex variable from a pair of linear integral equations. In the general case the region R is mapped within the unit circle in the ζ plane, and from a theorem of Harnack one may show that the formulation of the boundary problem in terms of the values at the boundary is equivalent to the interior problem.

The author's paper gives the formulation of plane strain and plane stress problems in two dimensional elasticity. For those regions which are circular or which may be mapped on the unit circle by a polynomial transformation, it is shown how the solutions may be obtained by a simple power series. The formulation of the system of integral equations is given for the more general case. Applications are made to a thermal stress problem in long cylindrical rods and to the deflection of nonisotropic thin plates which are loaded normal to the plane of the plate. In the latter example the generalized biharmonic plate equation requires the determination of two analytic functions of two complex variables $z_1=x+i\mu_1 y$ and $z_2=x+i\mu_2 y$ and their conjugates, where μ_i are the complex roots of the characteristic equation.

It is to be noted that the apparent simplicity of the formulation of the general problem does not insure an easy straight-forward method of obtaining practically useful solutions. This may arise if R is not simply connected or is infinite in extent or if singular load points occur on the boundary C, or if the mapping function offers difficulty in solving the system of integral equations. An excellent list of references is appended. *D. L. Holl* (Ames, Iowa).

Milne-Thomson, L. M. Consistency equations for the stresses in isotropic elastic and plastic materials. J. London Math. Soc. 17, 115–128 (1942). [MF 7319]

The necessary and sufficient compatibility equations for a symmetrical tensor Ψ to be capable of representing a strain tensor are obtained in the form (1) $\nabla\wedge\Psi\wedge\nabla=0$, where the notation $\mathbf{a}\wedge\mathbf{b}$ is used to represent the vector product of $\mathbf{a}$ and $\mathbf{b}$. Expansion theorems are proved for various tensors having the form of the left-hand side of (1). These are

applied to obtain the stress consistency equations when the stress and strain tensors are related by (a) Hooke's law, and by (b) Saint-Venant's hypothesis in the case of plastic deformation. The results can be applied to any coordinate system. For cases of plane deformation, the author still employs vector and dyadic notation, combined with the usual complex-number notation. Plane plastic and elastic deformation are both treated in detail. In the latter case, results in integrated form are obtained. *W. R. Sears.*

Prager, W. A new mathematical theory of plasticity. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 419–430 (1941). (English. Russian summary) [MF 7724]

The same paper has appeared in Rev. Fac. Sci. Univ. Istanbul (A) **5**, 215–226 (1941); these Rev. **3**, 32.

Sen, Bibhutibhusan. Stresses in an infinite strip due to an isolated couple acting at a point inside it. Bull. Calcutta Math. Soc. **34**, 45–51 (1942). [MF 7527]

The paper contains integral representations of the stresses produced in an infinite elastic strip by a couple acting at a point midway between the two edges. Approximate formulae for the nonvanishing stress components along the edges are given. *W. Prager.*

Myklestad, N. O. Two problems of thermal stress in the infinite solid. J. Appl. Mech. **9**, A-136–A-143 (1942). [MF 7125]

The first problem is that of determining the stresses set up in an isotropic infinite solid when part of it, bounded by an ellipsoid of revolution, is at a higher uniform temperature than the rest of the solid. In the second problem the region at a higher uniform temperature than the rest of the solid is a semi-infinite cylinder. The displacement at a point, with components u, v and w, is the gradient of a scalar function which in the problems of thermal stress under consideration is a solution of a differential equation having the form of Poisson's equation. The methods of the Newtonian potential theory are used to obtain the solutions of the problems proposed. *H. W. March* (Madison, Wis.).

Green, A. E. Stress systems in aelotropic plates. IV. Proc. Roy. Soc. London. Ser. A. **180**, 173–208 (1942). [MF 6928]

Solutions are obtained for problems of plane stress in an infinite aelotropic sheet containing a circular hole. The particular cases which are considered numerically are the stress distributions due to a uniform shear and bending moment and those due to a uniform normal pressure and to a uniform twist at the edge of the hole. The author also considers the stress distributions due to a normal pressure on the hole proportional to $\cos\theta$ and a tangential stress on the hole proportional to $\sin\theta$. It is found that taking into account the aelotropy of materials like spruce and oak leads to stress concentration factors materially different from those for an isotropic material. Complex variable methods are used throughout the paper. Reference should be made to a paper by S. Lechnickij [C. R. (Doklady) Acad. Sci. URSS (N.S.) **4**, 111–115 (1936)]. *E. Reissner.*

Müller, Emil. Rechteckige Platten, die an allen vier Seiten durch elastische Träger unterstützt sind. Ing.-Arch. **12**, 37–52 (1941). [MF 6861]

It is well known that a thin homogeneous rectangular plate can be solved by a method introduced by Levy when one pair of opposite edges are simply supported and for very general conditions at the other pair of edges. The author previously has solved this problem when two opposite edges are supported on elastic beams and the present solution deals with the case when all the edges are supported by elastic beams. An approximate solution is obtained by satisfying one of the boundary conditions at a finite number of points. A deflection function in the form of three series is employed,

$$\zeta(x, y)=\zeta_0+\sum_{1,3,5,\cdots}^{\infty} A_n\zeta_n+\sum_{1,3,5,\cdots}^{\infty} B_n\zeta_n,$$

which satisfies the moment condition at each edge and $\nabla^4\zeta_0=p/N$, $\nabla^4\zeta_n=\nabla^4\zeta_n=0$, with p equal to the unit normal load and N equal to the plate constant. The deflection ζ_0 is the known solution of the same simply supported plate problem. The remaining condition is the equivalence of the edge reaction of the plate and the load on the supporting beams. A finite number of the constants A_n and B_n are determined after using approximations in the known function ζ_0 for four types of symmetric and antisymmetric loads. A numerical example is appended in which the additional simplification is made that Poisson's ratio is zero.

D. L. Holl (Ames, Iowa).

Eschler, H. Beitrag zur elementaren Theorie des Querstosses auf Stäbe und Platten. Ing.-Arch. **12**, 31–37 (1941). [MF 6860]

The principles of conservation of energy and of momentum are employed to derive by elementary methods the maximum dynamical deflection of a simply supported beam or thin plate under the action of impact loads. The assumption is made that the dynamical wave surface is such that the ratio of the dynamical deflection to the static deflection at an arbitrary point is equivalent to the ratio of the deflections of the point of impact under moving load and static load. Then the same correspondence holds for velocities and an application of the principles of conservation introduces two factors which give the mass of the beam or plate an equivalent dynamical mass or "reduced mass." These "reduction factors" are dimensionless quantities which can be calculated when the static deflection surface is known for a unit load and for the given boundary conditions. The author treats three types of impact loads and concludes with a calculation of the reduction factors for a thin plate with the following edge conditions: (1) four edges simply supported, (2), (3) one pair of opposite edges simply supported and the other pair clamped or free and (4) all four edges clamped. *D. L. Holl* (Ames, Iowa).

Shulezhko, P. G. On the stability of thin elastic anisotropic plates of variable rigidity. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 139–150 (1942). (Russian. English summary) [MF 7745]

The work gives the expression for complete potential energy, the differential equation of the elastic surface of the plate, and the boundary conditions for anisotropic nonhomogeneous plates with variable rigidity, assuming that the middle surface of the plate is at the same time the surface of both elastic and geometric symmetry. It is shown that the boundary conditions of Kirchhoff, determined for the bending of plates, can not always be extended to buckling of plates. *Author's summary.*

Galerkin, B. G. On a stability problem in elastic systems. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 66–67 (1942). [MF 7612]

In this brief article the author presents (by reference to earlier publications) an eighth order partial differential equation which the displacement function φ must satisfy in thin circular shells of uniform thickness under symmetrical loading conditions. The radial, tangential and axial displacements (u, v, w) are derived from φ by linear combinations of fourth order partial derivatives of φ with respect to the circumferential and axial variables θ and z. It is stated that special cases of this general boundary problem include known solutions of various stability problems in axially symmetric rings and thin cylinders.

D. L. Holl (Ames, Iowa).

Brachkovsky, B. Z. Dynamic stability of elastic systems. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **6**, 87–88 (1942). (Russian. English summary) [MF 7737]

Nowoschilow, W. W. Einige Bemerkungen zur Theorie der Schalen. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] **5**, 375–382 (1941). (Russian. German summary) [MF 7721]

Die Arbeit befasst sich mit einigen Problemen der allgemeinen Schalentheorie.

In § 2 wird der Versuch einer Klassifizierung der existierenden Varianten der Schalentheorie je nach der Grösse der ihnen anhaftenden Mängel unternommen. Es wird die Tatsache hervorgehoben, dass viele Verfasser in ihren Formeln überflüssige kleine Glieder beibehalten, die eine Grösse derselben Ordnung darstellen, wie die Abweichung der Theorie als Ganzes. In §3 wird darauf hingewiessen, dass den Hypothesen Kirchhoffs-Love in der Schalentheorie an sich die Abweichung entspricht, die im Vergleich zur Einheit die Grösse der Ordnung t/R hat (t—die Schalenstärke, R—der geringste Radius der Krümmung der Mittelfläche), und dass dank diesem Umstande die Unterlagen der Hypothese nur beim Aufbau der Schalentheorie in ihrer ersten Approximation (d. h. der Theorie der dünnen Schalen) anwendbar sind. In §4 werden die Formeln der von E. Trefftz herrührenden successiven Theorie der dünnen Schalen angeführt. In §5 werden auf annäherndem Wege weitere drei Verhältnisse zwischen den Kräften und Momenten gefunden, mit deren Hilfe die Gleichungen der Theorie der dünnen Schalen in den Kräften und Momenten gelöst werden können. In §6 werden weitere Glieder angeführt, die bei der Prüfung der Probleme der Stabilität der dünnen Schalen in die Gleichungen aufgenommen werden müssen. *Author's summary.*

Pflüger, A. Spannungsverteilung in stabförmigen Membran-Kegelschalen. Z. Angew. Math. Mech. **22**, 99–116 (1942). [MF 7662]

In a thin walled hollow beam of variable cross section such as a tapered or truncated conical shell, the stress distribution is not given satisfactorily by the theory for an equivalent hollow cylindrical beam. In this paper the author applies the theory of general thin shells to the case of hollow tapered cantilever beam which experiences the axial, torsional, bending and shearing stresses which are induced by a general load applied at the free end. The equilibrium equations are set up for directions along lines of principal curvature and are made consistent with the displacements before the boundary conditions can be satisfied. In particular it is shown that eccentricity distance e of the center of twist for pure torsion does not coincide with the flexural shear center c, except in a uniform hollow cylinder or in a tapered beam with no flexural bending moment M. The general result may be stated as follows: The stress in a tapered hollow beam can be approximated by the known results of a hollow cylindrical beam if the shearing force Q is replaced by $Q-M/x$ and the twisting moment D is replaced by $D+eM/x$, where x is the distance of the section from the fixed end. *D. L. Holl.*

Miller, Harry. An analytical method for determining the flexibility of piping having two or more anchorages. J. Appl. Mech. **9**, A-165–A-170 (1942). [MF 7473]

The equilibrium of flexible pipes with bends in more than one plane is given by

$$\int_{s_0}^{s_1} \psi ds = 0, \quad \int_{s_1}^{s_2} [\psi \rho] ds = -\Delta\epsilon,$$

where ds is an element of pipe length at the vector position ρ, ψ is the rotation of pipe per unit length, $\Delta\epsilon$ the thermal expansion and the integral is carried out between two anchorages or between a given anchorage and various junctions. From the vector character of the integrands, the end reactions are determined from six simultaneous equations. *D. L. Holl* (Ames, Iowa).

Bull, Anders. Soil pressure distribution along flexible foundations. J. Franklin Inst. **233**, 559–580 (1942). [MF 6675]

Thoma, D. and Schilhansl, M. Spannungen und Formänderungen bei tordierten dünnwandigen Hohlzylindern mit kreisförmigem Ausschnitt. Luftfahrtforschung **19**, 210–214 (1942). [MF 7442]

According to an idea of R. V. Southwell [Proc. Roy. Soc. London. Ser. A. **163**, 337–355 (1937)] the torsion problem of a thin cylindrical shell can be approached by considering a rectangular plate under the action of uniform shear applied to the four edges and bending the distorted plate into a cylindrical form while the shear stresses remain operative. By bringing two opposite edges of the plate into contact and cementing them together, the application of external shearing stresses to these edges is rendered superfluous and a cylindrical tube subject to torsion is obtained. In the present paper this idea is used to estimate the effect of a circular cut-out on the stress distribution in a thin-walled cylindrical shell. In the case considered by Southwell there are no normal stresses acting perpendicular to the straight lines which become the generators of the cylinder. The equilibrium condition therefore is not affected by bending the plate into a cylindrical form. This is no longer true if the plate possesses a hole. Loads p, acting normal to the surface of the shell, must then be applied in order to restore equilibrium. Loading of the shell by $-p$ produces bending stresses which must be added to the stress system obtained from the shear distribution in a flat plate with a circular hole. If the diameter of the cut-out is small as compared with the diameter of the cylindrical shell, these additional stresses can be obtained approximately as the bending stresses produced in the flat plate by the transverse loads $-p$. A numerical example shows that the stress concentration due to the presence of the hole may become remarkably high. *W. Prager* (Providence, R. I.).